DUALITY FOR LIE-RINEHART ALGEBRAS AND THE MODULAR CLASS

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Abstract. A notion of homological duality for Lie-Rinehart algebras is studied which generalizes the ordinary duality in Lie algebra (co)homology and in the (co)homology of smooth manifolds. The duality isomorphisms can be given by cap products with suitable fundamental classes and hence may be taken to be natural in any reasonable sense. A precise notion of Poincaré duality, meaning that certain bilinear pairings over appropriate ground rings are nondegenerate, is then introduced, and various examples of Lie-Rinehart algebras are shown to satisfy Poincaré duality. Thereafter a certain intrinsic module introduced by Evens, Lu, and Weinstein for Lie algebroids is generalized to Lie-Rinehart algebras satisfying duality. This module determines a characteristic class, called the modular class of the Lie-Rinehart algebra; this class lies in an appropriately defined Picard group generalizing the abelian group of flat line bundles on a smooth manifold. A Poisson algebra having the requisite regularity properties determines a corresponding module for its Lie-Rinehart algebra and hence modular class whose square yields the module and characteristic class for its Lie-Rinehart algebra mentioned before. These concepts arise from abstraction from the notion of modular vector field for a smooth Poisson manifold. Finally, it is shown that the Poisson cohomology of certain Poisson algebras satisfies Poincaré duality.

Introduction

There are two ways of phrasing Poincaré duality in the real cohomology of a compact orientable smooth n-dimensional manifold W, by means of the natural bilinear (cup) pairing

$$(0.1)\qquad \mathrm{H}^j(W)\otimes_{\mathbb{R}}\mathrm{H}^{n-j}(W)\to\mathrm{H}^n(W),\quad 0\le j\le n,$$

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and via the natural map

$$\mathrm{H}^j(W) \longrightarrow \mathrm{H}_j(W), \quad 0 \leq j \leq n, \tag{0.2}$$

in singular homology which is obtained as the cap product with the fundamental class of W. Poincaré duality then amounts to the nondegeneracy of (0.1) or, equivalently, to (0.2) being an isomorphism. Both notions of Poincaré duality are equivalent since the universal coefficient map from $\mathrm{H}^j(W)$ to $\mathrm{Hom}(\mathrm{H}_j(W), \mathbb{R})$ is an isomorphism. Over more general rings, e. g. the integers, the appropriate way of phrasing Poincaré duality is in fact by means of the isomorphism (0.2) but this will not be important here. For our purposes, real cohomology of a smooth manifold will simply mean de Rham cohomology.

In the (co)homology of a finite dimensional Lie algebra over a field, Poincaré duality may likewise be phrased in two ways, which in fact formally correspond to the nondegeneracy of a pairing similar to (0.1) and to an isomorphism of the kind (0.2).

In this paper we shall abstract from both de Rham cohomology and Lie algebra (co)homology and study duality for Lie-Rinehart algebras (a definition will be given shortly). There are examples which do neither arise from the de Rham cohomology of a smooth manifold nor from ordinary Lie algebras, for example the Lie-Rinehart algebra of a foliation or that corresponding to a Poisson algebra, and these justify the general approach we offer here. With the appropriate notion of homology, the isomorphism (0.2) generalizes to arbitrary Lie-Rinehart algebras. We refer to this generalization simply as *duality* (or sometimes as *naive duality*). The bilinear pairing (0.1) may be generalized as well but the question whether the resulting pairing is nondegenerate (over an appropriate ring which does not necessarily coincide with the naive ground ring, see what is said below) is then rather delicate, and we do not know if the answer is always positive. We expect that, for a general Lie-Rinehart algebra, nondegeneracy reflects a certain regularity property, though, as it does for topological spaces where Poincaré duality singles out the *Poincaré complexes* (that is, cell complexes which satisfy the statement of the Poincaré duality theorem with respect to some fundamental class) and in particular manifolds. When nondegeneracy holds, we shall say that the Lie-Rinehart algebra satisfies *Poincaré duality*. Part of the paper is devoted to developing the necessary language to make this precise.

We now give an outline of the paper. Let R be a commutative ring and A a commutative R-algebra. An (R, A)-*Lie algebra* [21] is a Lie algebra L over R which acts on (the left of) A (by derivations) and is also an A-module satisfying suitable compatibility conditions which generalize the customary properties of the Lie algebra of smooth vector fields on a smooth manifold viewed as a module over its ring of smooth functions; these conditions read

$$\begin{aligned} [\alpha, a\beta] &= \alpha(a)\beta + a[\alpha, \beta] \\ (a\alpha)(b) &= a(\alpha(b)) \end{aligned} \tag{0.3}$$

where $a, b \in A$ and $\alpha, \beta \in L$. See our paper [15] for more details and history. When the ground ring R is not specified, we refer to the pair (A, L) as a *Lie-Rinehart* algebra to honor the fact that Rinehart [21] proved early non-trivial results about

these objects; in particular, he established a kind of Poincaré-Birkhoff-Witt theorem (which we reproduce in the proof of Theorem 2.10 below). Any (R, A)-Lie algebra L gives rise to a complex $\mathrm{Alt}_A(L, A)$ of alternating forms with the customary Cartan-Chevalley-Eilenberg differential [21] which generalizes the de Rham complex of a manifold *and* at the same time the complex computing Chevalley-Eilenberg Lie algebra cohomology, cf. e. g. [7]. More generally, there are notions of left- and right (A, L)-modules; given left- and right (A, L)-modules M and N, respectively, the cohomology $\mathrm{H}^*(L, M)$ and homology $\mathrm{H}_*(L, N)$ are defined in the usual way by $\mathrm{Ext}^*_U(A, M)$ and $\mathrm{Tor}^U_*(N, A)$, respectively, where $U = U(A, L)$ is the universal algebra for (A, L) (see Section 2 for details; it is the algebra of differential operators when A is the ring of smooth functions on a smooth manifold and L the Lie algebra of smooth vector fields). When L is projective as an A-module, the cohomology $\mathrm{H}^*(L, M)$ is computed by the complex $\mathrm{Alt}_A(L, M)$. When A and L are the algebra of smooth functions and Lie algebra of smooth vector fields, respectively, on a smooth manifold, the cohomology $\mathrm{H}^*(L, A)$ coincides with the de Rham cohomology of the manifold.

A Lie-Rinehart algebra (A, L) where, as an A-module, L is finitely generated projective of constant rank n will be referred to as a *duality Lie-Rinehart algebra of rank* n. For a general Lie-Rinehart algebra (A, L), the universal algebra $U = U(A, L)$ inherits structures of left- and right (A, L)-modules in the usual way. Suppose now that (A, L) is a duality Lie-Rinehart algebra of rank n. Consider the highest non-zero cohomology group $C_L = \mathrm{H}^{\mathrm{top}}(L, U) = \mathrm{H}^n(L, U)$, with reference to the left (A, L)-module structure on U. Under the construction of C_L, the right (A, L)-module structure on U remains free and induces a right (A, L)-module structure on C_L. In Section 2 we shall show the following:
(i) as right (A, L)-modules, C_L and $\mathrm{Hom}_A(\Lambda^n_A L, A)$ are isomorphic, the requisite right (A, L)-module structure on $\mathrm{Hom}_A(\Lambda^n_A L, A)$ being given by the negative of the Lie derivative; there are
(ii) natural isomorphisms

$$\mathrm{H}^k(L, M) \cong \mathrm{H}_{n-k}(L, C_L \otimes_A M) \tag{0.4}$$

for all non-negative integers k and all left (A, L)-modules M and, furthermore,
(iii) natural isomorphisms

$$\mathrm{H}_k(L, N) \cong \mathrm{H}^{n-k}(L, \mathrm{Hom}_A(C_L, N)) \tag{0.5}$$

for all non-negative integers k and all right (A, L)-modules N. See (2.11) below for details; the requisite left- and right (A, L)-module structures are there explained as well. A version of the isomorphism (0.4) may be found in [8 (5.2.2)].

In Section 3 we shall show that the duality isomorphisms (0.4) and (0.5) can be given by a cap product with a certain fundamental class in $\mathrm{H}_n(L, C_L)$; hence they are or may be taken to be natural in any reasonable sense. These versions of the isomorphisms (0.4) and (0.5) are the direct generalization of the isomorphism (0.2) spelled out above. For a general duality Lie-Rinehart algebra, they entail the existence of certain bilinear cohomology pairings which are not necessarily nondegenerate, though; see (3.10) and (7.13.1) below. Special cases of these pairings arising from certain Lie algebroids are given in [10].

In Section 4 we shall introduce a notion of Poincaré duality for a general duality Lie-Rinehart algebra, the terminology "Poincaré" referring to the nondegeneracy of certain cohomology pairings generalizing (0.1) above. A crucial ingredient is the notion of what we call a *trace*. For example, given a smooth n-dimensional manifold W, a trace for the Lie-Rinehart algebra $(A,L)=(C^\infty(W),\mathrm{Vect}(W))$ consists of the space $\mathcal{O}$ of (compactly supported) sections of the orientation bundle of W and an isomorphism t from $\mathrm{H}^n(L,\mathcal{O})$ ($\cong \mathrm{H}^n(W,\mathbb{R}_t)$ where $\mathbb{R}_t$ refers to the local system defined by the orientation bundle) to $\mathbb{R}$ (which is in fact given by integration). Our notion of 'trace' is similar to a corresponding one in the theory of Serre duality for the cohomology of projective sheaves on a projective scheme, cf. e. g. [14]. We shall prove that various duality Lie-Rinehart algebras, including the Lie-Rinehart algebra $(A,L)=(C^\infty(W),\mathrm{Vect}(W))$ where W is a smooth manifold and, more generally, the Lie-Rinehart algebra describing the infinitesimal structure of a fibre bundle with compact fibre, satisfy Poincaré duality for appropriate modules.

In Section 5 we shall show that, in a suitable sense, Poincaré duality is preserved under extensions of Lie-Rinehart algebras; see (5.3) for details. We then deduce that, cf. (5.4), every Lie-Rinehart algebra arising from a transitive Lie algebroid on a smooth manifold satisfies Poincaré duality.

In Section 6 we shall introduce a certain Picard group $\mathrm{Pic}^{\mathrm{flat}}(L,A)$ of projective rank one A-modules with a left (A,L)-module structure. This group could be thought of as a group of flat line bundles, the group structure being induced by the operation of tensor product over A. Thereafter we introduce a certain left (A,L)-module Q_L, for the special case where the algebra A is regular in the sense that, as an A-module, the (R,A)-Lie algebra $\mathrm{Der}(A)$ is finitely generated and projective. In this case, we denote the A-dual $\mathrm{Hom}_A(\Lambda^n_A\mathrm{Der}(A),A)$ of the highest non-zero exterior power $\Lambda^n_A\mathrm{Der}(A)$ of $\mathrm{Der}(A)$ by ω_A. For example, the algebra of smooth functions on a smooth manifold will be regular in this sense. The A-module ω_A may be identified with the highest non-zero exterior power of the module of formal differentials of A or of a suitable descendant thereof. In algebraic geometry, a closely related object is called the "canonical sheaf" and denoted by ω with an appropriate subscript whence the notation. When A is the algebra of smooth functions on a smooth manifold, ω_A is the space of sections of the highest non-zero exterior power of the cotangent bundle. In the general case, the negative of the operation of Lie derivative endows ω_A with a right (A,L)-module structure, and the module Q_L is then defined by $Q_L=\mathrm{Hom}_A(C_L,\omega_A)$; the customary diagonal action, with reference to the right (A,L)-module structures on C_L and ω_A, endows Q_L with a *left* (A,L)-module structure. When A is the algebra of smooth real functions on a smooth manifold and L arises from a foliation, the left (A,L)-module structure on Q_L amounts to the Bott connection for this foliation. For the special case of a Lie-Rinehart algebra arising from a Lie algebroid, the left (A,L)-module Q_L has been introduced in [10]. For a general duality Lie-Rinehart algebra (A,L), the class $[Q_L]\in\mathrm{Pic}^{\mathrm{flat}}(L,A)$ of the left (A,L)-module Q_L is then *characteristic* for (A,L). We refer to $[Q_L]$ as the *modular class* of (A,L). Our construction of this modular class is related with a similiar construction in [10] but substantially differs from the one in that paper and is more general. See Section 6 below for details. Our more formal approach involving Lie-Rinehart algebras over an arbitrary commutative algebra and the notion of duality explained in Sections 1 and 2 below and that of

Poincaré duality given in Section 4 place the results of [10] in their proper context, accordingly simplify the approach in that paper, and clarify some of the points related with existence of the modular class and the question whether or not certain cohomology pairings are nondegenerate, which have been left open there.

Finally, in Section 7 we shall apply our ideas and constructions to a Poisson algebra A which, as a commutative algebra, is assumed to be regular. Here the requisite (R, A)-Lie algebra $D_{\{\cdot,\cdot\}}$ is that arising from the formal differentials of A, introduced and studied in our paper [15], and the theory developed earlier applies. In particular, we have the corresponding left $(A, D_{\{\cdot,\cdot\}})$-module $Q_{D_{\{\cdot,\cdot\}}}$ at our disposal. From its Poisson structure, A also inherits a *right* $(A, D_{\{\cdot,\cdot\}})$-module structure; we denote the resulting right $(A, D_{\{\cdot,\cdot\}})$-module by $A_{\{\cdot,\cdot\}}$. This observation greatly simplifies the relevant calculations in [10] and allows for further generalization of the corresponding results in [10] and [22]. In fact, $C_{\{\cdot,\cdot\}}$ denoting the dualizing module for $D_{\{\cdot,\cdot\}}$ with its right $(A, D_{\{\cdot,\cdot\}})$-module structure, the diagonal action (cf. (2.3) below) endows the A-module $\mathrm{Hom}_A(C_{\{\cdot,\cdot\}}, A_{\{\cdot,\cdot\}})$ with a *left* $(A, D_{\{\cdot,\cdot\}})$-module structure. The left $(A, D_{\{\cdot,\cdot\}})$-module structure on $\mathrm{Hom}_A(C_{\{\cdot,\cdot\}}, A_{\{\cdot,\cdot\}})$ generalizes the modular vector field for a smooth Poisson manifold studied in [22] and elsewhere and, furthermore, offers some additional formal insight into this concept; some comments will be given after Lemma 7.8 below. Moreover, Theorem 7.9 will say that $Q_{D_{\{\cdot,\cdot\}}}$ and $\mathrm{Hom}_A(C_{\{\cdot,\cdot\}}, A_{\{\cdot,\cdot\}}) \otimes_A \mathrm{Hom}_A(C_{\{\cdot,\cdot\}}, A_{\{\cdot,\cdot\}})$ are isomorphic as left $(A, D_{\{\cdot,\cdot\}})$-modules. This generalizes a result in [10]. See (7.10) and (7.14) below. Moreover, the class $[\mathrm{Hom}_A(C_{\{\cdot,\cdot\}}, A_{\{\cdot,\cdot\}})] \in \mathrm{Pic}^{\mathrm{flat}}(D_{\{\cdot,\cdot\}}, A)$ of the left $(A, D_{\{\cdot,\cdot\}})$-module $\mathrm{Hom}_A(C_{\{\cdot,\cdot\}}, A_{\{\cdot,\cdot\}})$ is then *characteristic* for the Poisson algebra A. We refer to it as the *modular class* of A. In view of the relationship between $Q_{D_{\{\cdot,\cdot\}}}$ and $\mathrm{Hom}_A(C_{\{\cdot,\cdot\}}, A_{\{\cdot,\cdot\}})$ which we just mentioned, the modular class of the Lie-Rinehart algebra $(A, D_{\{\cdot,\cdot\}})$ is twice the modular class (or its square when we think of the group structure as being multiplicative) of the Poisson algebra A. This generalizes the corresponding observation spelled out in [10]. A related but different modular class for a Poisson manifold has been introduced in [22]. In the light of the notion of Poincaré duality to be introduced in Section 4, we then study the question of nondegeneracy of certain pairings in Poisson cohomology. Our approach differs substantially from that in [10]. Even in the special circumstances of [10], our pairings will not in general boil down to those in that paper. The requisite trace will involve the *left* $(A, D_{\{\cdot,\cdot\}})$-module $\mathrm{Hom}_A(C_{\{\cdot,\cdot\}}, A_{\{\cdot,\cdot\}})$ mentioned before. A special case, not so much interesting in itself but important for formal reasons, is that of the trivial Poisson structure: Given a smooth n-dimensional manifold, consider its algebra A of smooth functions, endowed with the trivial Poisson structure. The corresponding Poisson cohomology is just the exterior A-algebra $\Lambda_A V$ over the A-module V of smooth vector fields, and the corresponding pairing

$$\Lambda_A^k V \otimes_A \Lambda_A^{n-k} V \longrightarrow \Lambda_A^n V$$

is perfect over A; the algebra A here actually coincides with the algebra of Casimir functions. The nondegeneracy of this pairing cannot be phrased over the reals as ground ring. Our theory of Poincaré duality includes this example, see (7.15) below. Furthermore, at the end of Section 7, we give two non-trivial examples of nondegenerate bilinear pairings in Poisson cohomology, the pairings being defined

and nondegenerate over the subring of Casimir elements (and NOT over the naive ground ring, R). Again nondegeneracy seems to reflect a certain regularity property.

I am much indebted to A. Weinstein for having drawn my attention to his paper [22] and to [10]; in fact, the latter prompted me to write the present paper. I am grateful to J. Stasheff for a number of comments on earlier versions which helped improve the exposition, and to K. Mackenzie for discussions. Many thanks are due to the referee, for his careful reading of an earlier version of the paper, and for having prodded me to work out in detail the notion of Poincaré duality in terms of nondegenerate bilinear pairings.

It is a pleasure to dedicate this paper to my former teacher, Beno Eckmann, at the occasion of his 80th birthday in 1997 (when this paper was written). In my mathematical youth, when I was a student at the ETH (Zürich), duality groups were a fashionable topic, and the idea of duality which I then learnt from B. Eckmann (and R. Bieri), cf. e. g. [3], merged into the present paper; cf. also Remark 1.6 below.

1. Duality

Let R be a commutative ring, U an R-algebra, and let A be a left U-module. We shall consider the ordinary Ext- and Tor groups in the category of U-modules. Our convention is that (i) $\mathrm{Tor}_*^U(N, M)$ is defined for a right U-module N and a left U-module M, and that (ii) $\mathrm{Ext}_U^*(M_1, M_2)$, without further comment, refers to two left U-modules M_1 and M_2; occasionally we shall also consider $\mathrm{Ext}_U^*(N_1, N_2)$ for two right U-modules N_1 and N_2 but this will then always be indicated explicitly. We shall say that U and A *satisfy duality in dimension* n, with *dualizing module* C, provided there is a *right* U-module C such that one has *natural* isomorphisms

$$\mathrm{Ext}_U^k(A, M) \cong \mathrm{Tor}_{n-k}^U(C, M) \tag{1.1}$$

for all non-negative integers k and all left U-modules M. Likewise we shall say that U and A *satisfy inverse duality in dimension* n, with *dualizing module* D, provided there is a *right* U-module D such that one has *natural* isomorphisms

$$\mathrm{Tor}_k^U(N, A) \cong \mathrm{Ext}_U^{n-k}(D, N) \tag{1.2}$$

for all non-negative integers k and all right U-modules N, where $\mathrm{Ext}_U^*(\cdot,\cdot)$ is taken in the category of right U-modules. Here the convention is that $\mathrm{Ext}^*(\cdot,\cdot)$ and $\mathrm{Tor}_*(\cdot,\cdot)$ are zero in negative degrees.

The algebra U is a bimodule over itself in the usual way, and the groups $\mathrm{Ext}_U^*(A, U)$ (taken in the category of left U-modules) inherit right U-module structures from the right U-module structure on the second variable U which remains free when $\mathrm{Ext}_U^*(A, U)$ is taken. Recall that a left U-module M is said to have (U-)*dimension* n provided $\mathrm{Ext}_U^n(M, M')$ is non-zero for some left U-module M' and $\mathrm{Ext}_U^{k+n}(M, M'')$ is zero for every $k > 0$ and every left U-module M''; accordingly we can talk about the (U-)dimension of a right U-module. We shall say that a projective resolution $\varepsilon\colon K \to M$ of a left U-module M in the category of left U-modules is *finite of length* n provided each K_j is a finitely generated U-module and K_j is zero for $j > n$; likewise we can talk about *finite length* n projective resolutions in the category of right U-modules.

Proposition 1.3. *Suppose that U and A satisfy duality in dimension n, with dualizing module C. Then the following hold.*

(i) $\mathrm{Ext}^n_U(A,U) \cong C$ *as right U-modules, and* $\mathrm{Ext}^k_U(A,U) = 0$ *for* $k \neq n$, $k \geq 0$.

(ii) *As a left U-module, A has dimension n.*

(iii) *As a left U-module, A has a finite projective resolution of length n.*

Proof. Taking $M = U$ in (1.1), we find

$$\mathrm{Ext}^k_U(A,U) \cong \mathrm{Tor}^U_{n-k}(C,U) = \begin{cases} 0, & k \neq n \\ C, & k = n. \end{cases}$$

Hence (i) holds, and (ii) is clearly also true. Finally, we note that the functor $\mathrm{Tor}^U_*(C,\cdot)$ commutes with direct limits whence, by duality, the functor $\mathrm{Ext}^k_U(A,\cdot)$ commutes with direct limits for every k. In view of the corollary of Theorem 1 of [6], since the functor $\mathrm{Ext}^k_U(A,\cdot)$ commutes with direct limits and since A has U-dimension n, we conclude that A has a finite projective resolution of length n. $\square$

Proposition 1.4. *Suppose that the left U-module A has dimension n in such a way that* (i) *it has a finite projective resolution of length n, and that* (ii) $\mathrm{Ext}^k_U(A,U) = 0$ *for* $0 \leq k < n$. *Then U and A satisfy duality and inverse duality in dimension n, with (the same) dualizing module* $C = \mathrm{Ext}^n_U(A,U)$.

Proof. Let $\varepsilon\colon K \to A$ be a finite projective resolution of A of length n in the category of left U-modules. Then $K^* = \mathrm{Hom}_U(K,U)$ is a finite projective resolution of $C = \mathrm{Ext}^n_U(A,U)$ in the category of right U-modules. Hence, for every left U-module M, we have an isomorphism

$$\Phi\colon K^* \otimes_U M \longrightarrow \mathrm{Hom}_U(K,M) \tag{1.4.1}$$

which, for every non-negative integer k, induces

$$\phi\colon \mathrm{Tor}^U_{n-k}(C,M) \longrightarrow \mathrm{Ext}^k_U(A,M),$$

naturally in M. Likewise, for every right U-module N, we have an isomorphism

$$\Psi\colon N \otimes_U K \longrightarrow \mathrm{Hom}_U(K^*,N) \tag{1.4.2}$$

which, for every non-negative integer k, yields

$$\psi\colon \mathrm{Tor}^U_k(N,A) \longrightarrow \mathrm{Ext}^{n-k}_U(C,N),$$

naturally in N. $\square$

We note that, when duality holds in dimension $n > 0$, the dualizing module is obviously non-zero since otherwise the U-module A could not have stricly positive projective dimension; when duality holds in dimension $n = 0$, it amounts to $\mathrm{Hom}_U(A,M) \cong C \otimes_U M$, in particular, $\mathrm{Hom}_U(A,A) \cong C \otimes_U A$. Since $\mathrm{Hom}_U(A,A)$ contains at least a copy of the ground ring R, the dualizing module C cannot then be zero either.

REMARK 1.5. Let $\mathfrak{g}$ be a Lie algebra which, as a module over the ground ring R, is finitely generated and projective of (constant) rank n, let U be its universal algebra $U\mathfrak{g}$, and let $A = R$, with trivial $\mathfrak{g}$-module and hence trivial (unital) U-module structure. Then the ordinary duality isomorphisms in the homology and cohomology of $\mathfrak{g}$, cf. [7] (Exercise 15 on p. 288), are precisely of the kind (1.1) and (1.2), and the dualizing module $C = \mathrm{H}^n(\mathfrak{g}, U)$ has the form $\Lambda^n \mathfrak{g}^*$. See Example 4.5 below for details. In the next section we shall show that more generally, certain Lie-Rinehart algebras have similar duality properties.

REMARK 1.6. Let G be a discrete group, $A = R$, and $U = RG$, the group ring of G. The group G is called a *duality group* or *inverse duality group* provided there are natural isomorphisms of the kind (1.1) and (1.2), respectively, with C flat and D projective as R-modules. When G is a duality group, C is isomorphic to the highest non-zero cohomology group $\mathrm{H}^{\mathrm{top}}(G, RG)$ and when it is an inverse duality group, $D \cong \mathrm{H}^{\mathrm{top}}(G, RG)$. See [2] and [3] for details.

REMARK 1.7. It may be shown that, when U and A satisfy inverse duality in dimension n, with dualizing module D, the statements (i)-(iii) of Proposition 1.3 hold as well, with C being replaced by D. We refrain from spelling out details.

2. Lie-Rinehart algebras

We now suppose that A is a commutative R-algebra. Let L be an (R, A)-Lie algebra. Recall that its *universal algebra* $(U(A, L), \iota_L, \iota_A)$ is an R-algebra $U(A, L)$ together with a morphism $\iota_A: A \longrightarrow U(A, L)$ of R-algebras and a morphism $\iota_L: L \longrightarrow U(A, L)$ of Lie algebras over R having the properties

$$\iota_A(a)\iota_L(\alpha) = \iota_L(a\,\alpha), \quad \iota_L(\alpha)\iota_A(a) - \iota_A(a)\iota_L(\alpha) = \iota_A(\alpha(a)),$$

and $(U(A, L), \iota_L, \iota_A)$ is *universal* among triples (B, ϕ_L, ϕ_A) having these properties. More precisely: *Given*

(i) *another R-algebra B, viewed at the same time as a Lie algebra over R,*
(ii) *a morphism $\phi_L: L \longrightarrow B$ of Lie algebras over R, and*
(iii) *a morphism $\phi_A: A \longrightarrow B$ of R-algebras,*
so that, for $\alpha \in L, a \in A$,

$$\phi_A(a)\phi_L(\alpha) = \phi_L(a\,\alpha),$$
$$\phi_L(\alpha)\phi_A(a) - \phi_A(a)\phi_L(\alpha) = \phi_A(\alpha(a)),$$

there is a unique morphism $\Phi: U(A, L) \longrightarrow B$ of R-algebras so that $\Phi\,\iota_A = \phi_B$ and $\Phi\,\iota_L = \phi_L$. Often we shall simply write U instead of $(U(A, L), \iota_L, \iota_A)$. See (1.6) of [15] for more details. We only mention that U is determined up to isomorphism by the universal property as usual. For example, when A is the algebra of smooth functions on a smooth manifold B and L the Lie algebra of smooth vector fields on B, then $U(A, L)$ is the *algebra of (globally defined) differential operators on B.*

Suppose that A is endowed with the obvious left U-module structure induced by the left L-action on A. Our aim is to study the notions of duality and inverse duality for this special case.

Recall that an A-module M which is also a left L-module is called a *left* (A, L)-module provided

$$\alpha(ax) = \alpha(a)x + a\alpha(x) \tag{2.0.1}$$

$$(a\alpha)(x) = a(\alpha(x)) \tag{2.0.2}$$

where $a \in A,\ x \in M,\ \alpha \in L$. An A-module N which is also a right L-module, the action being written $(x, \alpha) \mapsto x\alpha$, is called a *right* (A, L)-module provided

$$(ax)\alpha = a(x\alpha) - \alpha(a)x \tag{2.0.3}$$

$$x(a\alpha) = a(x\alpha) - \alpha(a)x \tag{2.0.4}$$

where $a \in A,\ x \in M,\ \alpha \in L$. Then left- and right (A, L)-module structures correspond to left and right $U(A, L)$-module structures, respectively, and vice versa. The formula (2.0.4) might look somewhat puzzling at first glance; it is *not* the expected replica of (2.0.2). In view of the associativity law, there is only one consistent way to evaluate the expression $x(a\alpha)$ on the left-hand side of (2.0.4), though.

We now list a few facts related with the behaviour of (A, L)-modules under the operations "$\cdot \otimes_A \cdot$" and "$\mathrm{Hom}_A(\cdot, \cdot)$".

(2.1) Given two left (A, L)-modules M_1 and M_2, the customary formula

$$\alpha(m_1 \otimes m_2) = (\alpha m_1) \otimes m_2 + m_1 \otimes (\alpha m_2), \quad m_1 \in M_1,\ m_2 \in M_2,\ \alpha \in L,$$

endows their tensor product $M_1 \otimes_A M_2$ with a left (A, L)-module structure.

(2.2) Given two left (A, L)-modules M_1 and M_2, the customary formula

$$(\alpha\phi)(m) = \alpha(\phi m) - \phi(\alpha m), \quad m \in M_1,\ \alpha \in L,\ \phi \in \mathrm{Hom}_A(M_1, M_2),$$

endows the A-module $\mathrm{Hom}_A(M_1, M_2)$ with a left (A, L)-module structure.

(2.3) Given two right (A, L)-modules N_1 and N_2, the customary formula

$$(\alpha\phi)(n) = \phi(n\alpha) - (\phi n)\alpha, \quad n \in N_1,\ \alpha \in L,\ \phi \in \mathrm{Hom}_A(N_1, N_2),$$

endows the A-module $\mathrm{Hom}_A(N_1, N_2)$ with a left (!) (A, L)-module structure.

(2.4) Given left and right (A, L)-modules M and N, respectively, the formula

$$(n \otimes m)\alpha = (n\alpha) \otimes m - n \otimes (\alpha m), \quad m \in M,\ n \in N,\ \alpha \in L,$$

endows their tensor product $N \otimes_A M$ with a right (!) (A, L)-module structure.

(2.5) Given left and right (A, L)-modules M and N, respectively, the formula

$$(\phi\alpha)m = (\phi m)\alpha - \phi(\alpha m), \quad m \in M,\ \alpha \in L,\ \phi \in \mathrm{Hom}_A(M, N),$$

endows $\mathrm{Hom}_A(M, N)$ with a right (!) (A, L)-module structure.

We shall say that L *satisfies duality in dimension* n, with *dualizing module* C, provided U and A satisfy duality in dimension n, with dualizing module C; likewise we shall say that L *satisfies inverse duality in dimension* n, with *dualizing module* D, provided U and A satisfy inverse duality in dimension n, with dualizing module D.

Recall [21] that the classical complex of multilinear alternating forms extends from Lie algebras to Lie-Rinehart algebras; for a left (A, L)-module M, we denote by

$$(\mathrm{Alt}_A(L, M), d) \tag{2.6}$$

the resulting chain complex (over R) of M-valued A-multilinear alternating forms, where d is the corresponding Cartan-Chevalley-Eilenberg operator. We only note that the fact that this operator sends A-multilinear forms to A-multilinear forms is not obvious and requires proof; see [21] and what is said below. Further, the ordinary constructions of *contraction*

$$i\colon L \otimes_R \mathrm{Alt}_A(L, M) \longrightarrow \mathrm{Alt}_A(L, M) \tag{2.6.1}$$

and *Lie derivative*

$$\lambda\colon L \otimes_R (\mathrm{Alt}_A(L, M), d) \longrightarrow (\mathrm{Alt}_A(L, M), d) \tag{2.6.2}$$

carry over as well [21]; for $\alpha \in L$, we write

$$i_\alpha\colon \mathrm{Alt}_A(L, M) \longrightarrow \mathrm{Alt}_A(L, M)$$

and

$$\lambda_\alpha\colon (\mathrm{Alt}_A(L, M), d) \longrightarrow (\mathrm{Alt}_A(L, M), d),$$

respectively. The operation of Lie derivative endows $(\mathrm{Alt}_A(L, M), d)$ with a *left* L-module structure (*not* with that of a left (A, L)-module) and, as is common for ordinary Lie algebras, the negative of the Lie derivative yields a *right* L-module structure on $(\mathrm{Alt}_A(L, M), d)$ (*not* that of a right (A, L)-module). These operations satisfy the customary formulas

$$\lambda_{a\alpha}\omega = a\lambda_\alpha\omega + (da) \cup (i_\alpha\omega), \tag{2.7.1}$$

$$\lambda_\alpha = di_\alpha + i_\alpha d, \tag{2.7.2}$$

$$i_\alpha(\omega_1 \cup \omega_2) = i_\alpha(\omega_1) \cup \omega_2 + (-1)^{\deg(\omega_1)}\omega_1 \cup i_\alpha(\omega_2), \tag{2.7.3}$$

where $\alpha \in L$, $a \in A$, $\omega, \omega_1, \omega_2 \in \mathrm{Alt}_A(L, M)$.

Recall that, for a finitely generated projective A-module, its rank to be constant means that it is the same for every prime ideal of A. For example, when A is the algebra of smooth functions on a smooth manifold W and L the $(\mathbb{R}, A)$-Lie algebra of smooth vector fields on W, the requirement that L be of constant rank amounts to the connectedness of W.

Proposition 2.8. *Suppose that, as an A-module, the (R, A)-Lie algebra L is finitely generated projective of constant rank n. For every left (A, L)-module M, the formula*

$$\phi\alpha = -\lambda_\alpha(\phi), \quad \phi \in \mathrm{Hom}_A(\Lambda^n_A L, M),\ \alpha \in L, \tag{2.8.1}$$

then endows $\mathrm{Hom}_A(\Lambda^n_A L, M)$ *with a right* (!) (A, L)*-module structure.*

Proof. For $a \in A$, $\alpha \in L$, $\phi \in \mathrm{Hom}_A(\Lambda^n_A L, M) = \mathrm{Alt}^n_A(L, M)$, in view of (2.7.1) and (2.7.3), we have

$$\lambda_{a\alpha}\phi = a\lambda_\alpha\omega + (da) \cup (i_\alpha\phi)$$

and

$$i_\alpha((da) \cup \phi) = (i_\alpha(da)) \cup \phi - (da) \cup (i_\alpha\phi) = \alpha(a)\phi - (da) \cup (i_\alpha\phi).$$

Since ϕ is in the top dimension, $(da) \cup \phi$ is zero whence

$$(da) \cup (i_\alpha \phi) = \alpha(a)\phi.$$

Thus $\lambda_{a\alpha}\phi = a\lambda_\alpha \omega + \alpha(a)\phi$ and thence $\phi(a\alpha) = a(\phi\alpha) - \alpha(a)\phi$. Consequently (2.8.1) yields a right (A, L)-module structure on $\mathrm{Hom}_A(\Lambda^n_A L, M)$ as asserted. □

A special case of this proposition is known from the theory of D-modules, cf. (VI.3.2) and (VI.3.3) on pp. 226/227 of [5].

Under appropriate circumstances, the chain complex (2.6) computes the cohomology of the (R, A)-Lie algebra L with values in M. To recall what this means, we reproduce briefly the Rinehart complex for (A, L); in the next section, we shall further exploit this Rinehart complex and in particular deduce descriptions of the duality isomorphisms (to be given in (2.11) below) in terms of appropriate cup- and cap products: Let $\Lambda_A L$ be the exterior A-algebra of L, and consider the graded left $U(A, L)$-module $U(A, L) \otimes_A \Lambda_A L$ where A acts on $U(A, L)$ from the right by means of the canonical map from A to $U(A, L)$. For $u \in U(A, L)$ and $\alpha_1, \ldots, \alpha_n \in L$, let

$$\begin{aligned} & d(u \otimes_A (\alpha_1 \wedge \cdots \wedge \alpha_n)) \\ (2.9.1) \qquad & = \sum_{i=1}^n (-1)^{(i-1)} u\alpha_i \otimes_A (\alpha_1 \wedge \ldots \widehat{\alpha_i} \ldots \wedge \alpha_n) \\ & \quad + \sum_{j<k} (-1)^{(j+k)} u \otimes_A ([\alpha_j, \alpha_k] \wedge \alpha_1 \wedge \ldots \widehat{\alpha_j} \ldots \widehat{\alpha_k} \ldots \wedge \alpha_n). \end{aligned}$$

Rinehart [21] has proved that this yields an $U(A, L)$-linear differential

$$d: U(A, L) \otimes_A \Lambda_A L \longrightarrow U(A, L) \otimes_A \Lambda_A L,$$

that is, $dd = 0$. The non-trivial fact to be verified here is that, for every $u \in U(A, L)$, $a \in A$, and $\alpha_1, \ldots, \alpha_n \in L$,

$$d((ua) \otimes (\alpha_1 \wedge \cdots \wedge \alpha_n)) = d(u \otimes ((a\alpha_1) \wedge \cdots \wedge \alpha_n)) = \cdots = d(u \otimes (\alpha_1 \wedge \cdots \wedge (a\alpha_n))).$$

We only mention that, separately, the two summands on the right-hand side of (2.9.1) are *not* even well defined, and only their sum is A-multilinear in the variables $u, \alpha_1, \ldots, \alpha_n$. We refer to the resulting chain complex

$$(2.9.2) \qquad K(A, L) = (U(A, L) \otimes_A \Lambda_A L, d)$$

as the *Rinehart complex* for (A, L). Rinehart has also proved that, when L is projective as an A-module, $K(A, L)$ is in fact a projective resolution of A in the category of left $U(A, L)$-modules.

We now recall that, for a general (R, A)-Lie algebra L and for left and right (A, L)-modules M and N, respectively, the homology and cohomology of L are defined by

$$(2.9.3) \qquad \mathrm{H}^*(L, M) = \mathrm{Ext}^*_U(A, M) \quad \text{and} \quad \mathrm{H}_*(L, N) = \mathrm{Tor}^U_*(N, A).$$

When L is projective as an A-module, its homology and cohomology may thus be computed from $K(A,L)$; hence, given a left (A,L)-module M, the cohomology $\mathrm{H}^*(L,M)$ may be obtained as that of the chain complex

$$(\mathrm{Alt}_A(L,M),d) = \mathrm{Hom}_U(K(A,L),M) \tag{2.9.4}$$

and a similar statement can be made for the homology $\mathrm{H}_*(L,N)$ of L with values in a right (A,L)-module N. Notice that the description (2.9.4) of $(\mathrm{Alt}_A(L,M),d)$ justifies in particular the earlier claim (which may also be verified directly) that the Cartan-Chevalley-Eilenberg operator sends A-multilinear forms to A-multilinear forms. (The notation "d" is slightly abused here.) For an ordinary Lie algebra $\mathfrak{g}$ over R, viewed as an (R,A)-Lie algebra with trivial L-action on $A=R$, the Rinehart complex $K(R,\mathfrak{g})$ comes down to the standard *Koszul complex* which we denote by $K_R\mathfrak{g}$ (or Koszul resolution of R when $\mathfrak{g}$ is projective as an R-module) familiar in ordinary Lie algebra (co)homology.

When the (R,A)-Lie algebra L is finitely generated and projective of constant rank n as an A-module, its Rinehart complexe $K(A,L)$ is manifestly a finite projective resolution of A of length n in the category of left $U(A,L)$-modules, and the highest non-zero term $K_n(A,L)$ of the resolution has the form $U(A,L)\otimes_A \Lambda^n_A L$.

Theorem 2.10. *An (R,A)-Lie algebra L which, as an A-module, is finitely generated and projective of constant rank n, satisfies duality and inverse duality in dimension n, with dualizing module*

$$C_L = \Lambda^n_A L^* = \mathrm{Hom}_{\mathrm{A}}(\Lambda^n_A L, A),$$

the requisite isomorphism between C_L and $\mathrm{H}^n(L,U)$ being induced by the injection of $\mathrm{Hom}_{\mathrm{A}}(\Lambda^n_A L, U)$ into

$$\mathrm{Hom}_{\mathrm{A}}(\Lambda^n_A L, U) \cong \mathrm{Hom}_{\mathrm{U}}(K_n(A,L),U)$$

coming from the canonical inclusion of A into $U=U(A,L)$; here $\mathrm{Hom}_{\mathrm{A}}(\Lambda^n_A L, A)$ is viewed endowed with the right (A,L)-module structure (2.8.1).

Proof. We show that the conditions of Proposition 1.4 are met. To this end, using the fact that L is projective as an A-module, we exploit the Poincaré-Birkhoff-Witt theorem for Lie-Rinehart algebras [21] and consider the customary filtration of $U=U(A,L)$ coming from powers of L and having as associated graded algebra $\mathrm{E}^0(U)$ the symmetric algebra $S_A L$ on L in the category of A-modules. This filtration induces a spectral sequence having $\mathrm{E}_1 = \mathrm{Ext}^*_{S_A L}(A, S_A L)$ and converging to $\mathrm{Ext}^*_U(A,U)$. But $\mathrm{Ext}^k_{S_A L}(A, S_A L) = 0$ for $0 \le k < n$, and the injection of $\mathrm{Hom}_{\mathrm{A}}(\Lambda^n_A L, A)$ into $\mathrm{Hom}_{\mathrm{A}}(\Lambda^n_A L, S_A L)$ coming from the canonical inclusion of A into $S_A L$ induces an isomorphism from $\mathrm{Hom}_{\mathrm{A}}(\Lambda^n_A L, A)$ onto $\mathrm{Ext}^n_{S_A L}(A, S_A L)$ whence the spectral sequence has $\mathrm{E}_1 = \mathrm{E}_\infty$. Hence, for $0 \le k < n$, $\mathrm{Ext}^k_U(A,U) = 0$, and the injection of $\mathrm{Hom}_{\mathrm{A}}(\Lambda^n_A L, A)$ into $\mathrm{Hom}_{\mathrm{A}}(\Lambda^n_A L, U)$ coming from the canonical inclusion of A into U induces an isomorphism

$$C_L = \mathrm{Hom}_{\mathrm{A}}(\Lambda^n_A L, A) \to \mathrm{H}^n(L,U). \tag{2.10.1}$$

To see that this is an isomorphism of right (A, L)-modules of the asserted kind, let $\alpha, \alpha_1, \dots, \alpha_n \in L$ and $\phi \in \mathrm{Hom_A}(\Lambda^n_A L, A)$ so that $\phi(\alpha_1, \dots, \alpha_n) \in A$; a straightforward calculation shows that, then,

$$\begin{aligned}(\phi\alpha)(\alpha_1, \dots, \alpha_n) + (\lambda_\alpha\phi)(\alpha_1, \dots, \alpha_n) &= -\alpha(\phi(\alpha_1, \dots, \alpha_n)) + \alpha \cdot (\phi(\alpha_1, \dots, \alpha_n)) \\ &= (\phi(\alpha_1, \dots, \alpha_n)) \cdot \alpha;\end{aligned}$$

here "$\cdot$" refers to the product in $U = U(A, L)$, $\phi\alpha \in \mathrm{Hom_A}(\Lambda^n_A L, A)$ denotes the result of the operation on $\phi \in \mathrm{Hom_A}(\Lambda^n_A L, A)$ with $\alpha \in L$ from the right, $\alpha(\phi(\alpha_1, \dots, \alpha_n)) \in A$ that of the operation on $\phi(\alpha_1, \dots, \alpha_n) \in A$ with α from the left, and the equality sign means 'identity in $\mathrm{Hom_A}(\Lambda^n_A L, U)$', $\mathrm{Hom_A}(\Lambda^n_A L, A)$ being viewed as a subspace thereof via the obvious injection of A into U. Since we are in the top dimension, the operation λ_α equals di_α whence

$$\lambda_\alpha\phi = di_\alpha\phi$$

is a coboundary. Thus the cocycles $\phi\alpha$ and

$$(\alpha_1, \dots, \alpha_n) \mapsto (\phi(\alpha_1, \dots, \alpha_n)) \cdot \alpha$$

are cohomologous. Hence (2.10.1) is even an isomorphism of right (A, L)-modules. In view of Proposition 1.4, this proves the claim. $\square$

Theorem 2.10 suggests the following definition: An (R, A)-Lie algebra L which, as an A-module, is finitely generated and projective of constant rank, will henceforth be referred to as a *duality* (R, A)-*Lie algebra* and the pair (A, L) will be called a *duality Lie-Rinehart algebra*. The rank of a duality (R, A)-Lie algebra L, viewed as an A-module, will be referred to as the *rank* of L and, likewise, we shall talk about the *rank* of a duality Lie-Rinehart algebra.

Corollary 2.11. *Given a duality (R, A)-Lie algebra L of rank n, there are natural isomorphisms*

$$\phi\colon \mathrm{H}_k(L, C_L \otimes_A M) \longrightarrow \mathrm{H}^{n-k}(L, M) \tag{2.11.1}$$

for all non-negative integers k and all left (A, L)-modules M and, furthermore, natural isomorphisms

$$\psi\colon \mathrm{H}_k(L, N) \longrightarrow \mathrm{H}^{n-k}(L, \mathrm{Hom}_A(C_L, N)) \tag{2.11.2}$$

for all non-negative integers k and all right (A, L)-modules N.

Here $C_L \otimes_A M$ and $\mathrm{Hom}_A(C_L, N)$ carry the corresponding right- and left (A, L)-module structures explained in (2.4) and (2.3), respectively.

A duality isomorphism of the kind (2.11.1) may be found in [8 (5.2.2)]. The additional information provided for by Theorem 2.10 does not seem to be in the literature, though. Occasionally we refer to duality isomorphisms of the kind (2.11.1) and (2.11.2) as *naive duality*.

Example 2.12. Let $\mathfrak{g}$ be a Lie algebra over R which, as an R-module, is supposed to be finitely generated and projective of constant rank n (say), let A be a commutative

R-algebra, and suppose that $\mathfrak{g}$ acts on A by derivations. In view of the defining properties (0.3) for an (R, A)-Lie algebra, the Lie bracket on $\mathfrak{g}$ and the $\mathfrak{g}$-action on A induce a bracket on $L = A \otimes_R \mathfrak{g}$ which, together with the obvious left A-module structure on L, turns L into an (R, A)-Lie algebra. As an A-module, the dualizing module C_L of L is plainly isomorphic to $A \otimes_R \Lambda^n \mathfrak{g}^*$. Moreover, the universal algebra $U(A, L)$ may be written in the form $A \otimes_R U\mathfrak{g}$, and its algebra structure is given by

$$(a \otimes_R 1)(1 \otimes_R x) = a \otimes_R x, \quad (1 \otimes_R x)(a \otimes_R 1) = a \otimes_R x + (x(a)) \otimes_R 1,$$

where $a \in A$ and $x \in \mathfrak{g}$. Further, the duality isomorphisms of the kind (2.11.1) and (2.11.2) obtain. Now, $(R, \mathfrak{g})$ being considered as a Lie-Rinehart algebra as well, with trivial $\mathfrak{g}$-action on R (which coincides with A in this case), we denote the corresponding dualizing module by $C_{\mathfrak{g}}$ ($\cong \Lambda^n \mathfrak{g}^*$). When A is different from R (and comes with a non-trivial $\mathfrak{g}$-action), as an A-module, C_L is plainly isomorphic to $A \otimes_R C_{\mathfrak{g}}$, and on this isomorphic image of C_L, the right (A, L)-module structure is induced by the obvious right $\mathfrak{g}$-module structure on $A \otimes_R C_{\mathfrak{g}}$ and may thus be described by the formula

$$(2.12.1) \qquad (b \otimes_R \phi)(a \otimes_R x) = -(ax(b)) \otimes_R \phi - a \otimes_R (\lambda_x \phi) - (x(a)b) \otimes_R \phi;$$

here $a, b \in A$, $x \in \mathfrak{g}$, $\phi \in C_{\mathfrak{g}} \cong \Lambda^n \mathfrak{g}^*$ and, for $y \in \Lambda^n \mathfrak{g}$, the value $(\lambda_x \phi)(y)$ is given by $(\lambda_x \phi)(y) = -\phi(x(y))$. The formula (2.12.1) may look a bit odd but is forced by (2.0.3) and (2.0.4). The duality isomorphisms for (A, L) now boil down to those for the ordinary Lie algebra $\mathfrak{g}$. In fact, given left- and right (A, L)-modules M and N, respectively, the duality isomorphisms (2.11.1) and (2.11.2) come down to

$$(2.12.2) \qquad \mathrm{H}_k(L, C_L \otimes_A M) \cong \mathrm{H}_k(\mathfrak{g}, C_{\mathfrak{g}} \otimes_R M) \xrightarrow{\phi} \mathrm{H}^{n-k}(\mathfrak{g}, M) \cong \mathrm{H}^{n-k}(L, M)$$

and

$$(2.12.3) \quad \mathrm{H}_k(L, N) \cong \mathrm{H}_k(\mathfrak{g}, N) \xrightarrow{\psi} \mathrm{H}^{n-k}(\mathfrak{g}, \mathrm{Hom}_R(C_{\mathfrak{g}}, N)) \cong \mathrm{H}^{n-k}(L, \mathrm{Hom}_A(C_L, N)),$$

respectively. Cf. Remark 1.5 above. For later reference we mention that, given a left $\mathfrak{g}$-module $\mathfrak{m}$, the induced A-module $A \otimes_R \mathfrak{m}$ inherits an obvious left (A, L)-module structure given by the formula

$$(2.12.4) \quad (a \otimes_R x)(b \otimes_R y) = (ax(b)) \otimes_R y + (ab) \otimes_R (xy), \quad a, b \in A, \ x \in \mathfrak{g}, \ y \in \mathfrak{m},$$

which is in fact forced by (2.0.1) and (2.0.2). Furthermore, given any left (A, L)-module M, the formula

$$(2.12.5) \qquad y(a \otimes_R x) = -(a \otimes_R x)y - (x(a))y, \quad a \in A, \ x \in \mathfrak{g}, \ y \in M,$$

yields a right (A, L)-module structure on M; this is forced by (2.0.3) and (2.0.4). For a general Lie-Rinehart algebra, we cannot naively pass from left- to right (A, L)-module structures in such a naive fashion, though.

3. Multiplicative structures

In the definition of duality, we did not require that the duality isomorphisms commute with connecting homomorphisms nor with maps induced by morphisms in the (R,A)-Lie algebra argument. However, Theorem 3.7 below shows that duality isomorphisms can be given by a cap-product, and these are, of course, natural in any reasonable sense.

As before, let L be an (R,A)-Lie algebra. Recall from Section 1 of [15], cf. also (1.6) in [17], that, given a pairing $M_1 \otimes_A M_2 \to M$ of left (A,L)-modules, the standard shuffle multiplication of alternating maps induces a pairing

$$\cup\colon \mathrm{Alt}_A(L,M_1) \otimes_R \mathrm{Alt}_A(L,M_2) \longrightarrow \mathrm{Alt}_A(L,M) \tag{3.1}$$

of chain complexes over the ground ring R (see (1.5') in [15]). When L is projective as an A-module so that the homology of $\mathrm{Alt}_A(L,M)$ etc. gives $\mathrm{H}^*(L,M)$ etc., the pairing (3.1) induces a pairing in the cohomology of L generalizing the ordinary *cup* pairing in Lie algebra cohomology and we refer to the resulting pairing as *cup pairing* as well, written

$$\cup\colon \mathrm{H}^*(L,M_1) \otimes_R \mathrm{H}^*(L,M_2) \longrightarrow \mathrm{H}^*(L,M) \tag{3.2}$$

(with an abuse of the notation $\cup$). This pairing also generalizes the ordinary multiplicative structure in de Rham cohomology.

For intelligibility, we recall an explicit description; it is not completely simple because the distinction between graded A-objects and differential graded R-modules due to the in general non-trivial action of L on A persists throughout: The ordinary diagonal map

$$\Delta\colon \Lambda_R L \longrightarrow \Lambda_R L \otimes \Lambda_R L \tag{3.1.1}$$

determined by

$$\Delta(v) = v \otimes 1 + 1 \otimes v, \quad v \in L, \tag{3.1.2}$$

makes the graded exterior R-algebra $\Lambda_R L$ into a graded commutative and graded cocommutative R-Hopf algebra and hence in particular endows $\Lambda_R L$ with a graded cocommutative coalgebra structure; see e. g. Mac Lane [20] for details. This diagonal map is also referred to as *shuffle coproduct* or *shuffle diagonal*. Since the property of being a Hopf algebra implies in particular that its diagonal map is multiplicative, the assignment (3.1.2) in fact completely determines (3.1.1). Explicitly, given $x_1,\dots,x_{p+q} \in L$, the value $\Delta(x_1 \wedge \dots \wedge x_{p+q})$ in $\Lambda_R L$ is given by the formula

$$\begin{aligned}\Delta(&x_1 \wedge \dots \wedge x_{p+q}) \\ &= \sum_\sigma \mathrm{sign}(\sigma)(x_{\sigma(1)} \wedge \dots \wedge x_{\sigma(p)}) \otimes (x_{\sigma(p+1)} \wedge \dots \wedge x_{\sigma(p+q)}),\end{aligned} \tag{3.1.3}$$

where σ runs through (p,q)-shuffles and where $\mathrm{sign}(\sigma)$ refers to the sign of σ. Formally the same construction, but in the category of A-modules rather than in that of R-modules, yields a diagonal map

$$\Delta_A\colon \Lambda_A L \longrightarrow \Lambda_A L \otimes_A \Lambda_A L \tag{3.1.4}$$

on (on the exterior A-algebra) $\Lambda_A L$ determined by

$$\Delta_A(v) = v \otimes_A 1 + 1 \otimes_A v, \quad v \in L, \tag{3.1.5}$$

and this diagonal map makes the graded exterior A-algebra $\Lambda_A L$ into a graded commutative and graded cocommutative A-Hopf algebra and hence in particular endows $\Lambda_A L$ with a graded cocommutative A-coalgebra structure. Moreover the two diagonal maps (3.1.1) and (3.1.4) are compatible in the sense that the diagram

$$\begin{array}{ccc} \Lambda_R L & \xrightarrow{\Delta} & \Lambda_R L \otimes \Lambda_R L \\ \big\downarrow & & \big\downarrow \\ \Lambda_A L & \xrightarrow{\Delta_A} & \Lambda_A L \otimes_A \Lambda_A L \end{array} \tag{3.1.6}$$

is commutative, the unlabelled arrows being the obvious maps. Dualization now yields the pairing (3.1): Given $\alpha \in \mathrm{Alt}^p_A(L, M_1)$ and $\beta \in \mathrm{Alt}^q_A(L, M_2)$ and, furthermore, arbitrary $x_1, \ldots, x_{p+q} \in L$, the value $(\alpha \cup \beta)(x_1, \ldots, x_{p+q})$ in $\mathrm{Alt}^{p+q}_A(L, M)$ is given by the explicit expression

$$\begin{aligned} &(\alpha \cup \beta)(x_1, \ldots, x_{p+q}) \\ &\qquad = (-1)^{|\alpha||\beta|} \sum_\sigma \mathrm{sign}(\sigma) \mu_A(\alpha(x_{\sigma(1)}, \ldots, x_{\sigma(p)}) \otimes_A \beta(x_{\sigma(p+1)}, \ldots, x_{\sigma(p+q)})), \end{aligned}$$

where $\mu_A \colon M_1 \otimes_A M_2 \to M$ denotes the given (A, L)-module pairing from $M_1 \otimes_A M_2$ to M. Without the differentials, we would also obtain a pairing

$$\mathrm{Alt}_A(L, M_1) \otimes_A \mathrm{Alt}_A(L, M_2) \longrightarrow \mathrm{Alt}_A(L, M)$$

of graded A-modules but for reasons already explained our pairing of primary interest is (3.1).

Likewise, for left- and right (A, L)-modules M and N, respectively, the standard construction still yields a cap-pairing. Notice that, for reasons already hinted at, we must be a bit circumspect here: the standard construction of multiplicative structures in e. g. the cohomology of a Lie algebra in the ordinary sense (over the ground ring R) involves the diagonal map on its universal algebra; under our more general circumstances of a general (R, A)-Lie algebra L the universal algebra $U(A, L)$ will not in general inherit a Hopf algebra structure (over R) unless L is a Lie algebra over A in the ordinary sense, that is, when L acts trivially on A and A coincides with the ground ring R. In fact, when the algebra A is different from the ground ring and L acts non-trivially on A, the question of Hopf algebra structure on $U(A, L)$ is only well posed when A itself comes with a Hopf algebra structure compatible with the L-action; this happens to be the case, for example, when A is the algebra of algebraic functions on an algebraic group, e. g. on a compact Lie group over the reals, and when L is the Lie algebra of derivations of such an A. We refrain from spelling out details, since we shall not need them.

Theorem 3.3. *When L is projective as an A-module, for left- and right (A,L)-modules M and N, respectively, the customary formula involving the requisite shuffle map yields a cap pairing*

$$\cap\colon \mathrm{H}_\ell(L,N)\otimes_R \mathrm{H}^k(L,M) \longrightarrow \mathrm{H}_{\ell-k}(L,N\otimes_A M) \tag{3.3.1}$$

which is natural in terms of the data.

To prepare for the proof, we recall the notion of cap product, in a form tailored to our purposes: Let Λ be a (differential) graded R-coalgebra, and let M and N be R-modules (we could allow M and N to be R-chain complexes but we do not need this greater generality). Given $\phi \in \mathrm{Hom}_R(\Lambda, M)$, the morphism

$$\cdot \cap \phi\colon N\otimes_R \Lambda \longrightarrow N\otimes_R M\otimes_R \Lambda$$

is defined as the composite

$$N\otimes_R \Lambda \xrightarrow{N\otimes_R \Delta} N\otimes_R \Lambda\otimes_R \Lambda \xrightarrow{N\otimes_R\phi\otimes_R\Lambda} N\otimes_R M\otimes_R \Lambda,$$

where the identity morphism on an object is denoted by the same symbol as that object. The resulting pairing

$$\cap\colon N\otimes_R \Lambda\otimes_R \mathrm{Hom}_R(\Lambda,M) \longrightarrow N\otimes_R M\otimes_R \Lambda \tag{3.3.2}$$

is a version of the ordinary *cap pairing* in differential homological algebra. See e. g. (2.3) in [13]. We note that, in order to get differentials right (whenever differentials come into play), the Eilenberg-Koszul convention should systematically be in force: Whenever two graded objects a and b (say) are interchanged, the sign $(-1)^{|a||b|}$ should be added. In particular, for an ordinary R-Lie algebra $\mathfrak g$, given left- and right $\mathfrak g$-modules M and N, respectively, taking $\Lambda = \Lambda_R\mathfrak g$, with the shuffle diagonal (3.1.1), the pairing (3.3.2), viewed as one of graded R-modules, may also be written

$$\cap\colon (N\otimes_{U\mathfrak g} K_R\mathfrak g)\otimes_R \mathrm{Alt}_R(\mathfrak g, M) \longrightarrow (N\otimes_R M)\otimes_{U\mathfrak g} K_R\mathfrak g$$

where $K_R\mathfrak g$ denotes the Koszul complex for $\mathfrak g$ in the category of R-modules, cf. what has been said in the previous section ($K_R\mathfrak g$ coincides with the Rinehart complex $K(R,\mathfrak g)$, cf. (2.9.2)). In this form, the pairing is actually compatible with the requisite differentials (coming from the Koszul complex $K_R\mathfrak g$). When $\mathfrak g$ is projective as an R-module, this pairing computes the customary cap pairing

$$\cap\colon \mathrm{H}_\ell(\mathfrak g,N)\otimes_R \mathrm{H}^k(\mathfrak g,M) \longrightarrow \mathrm{H}_{\ell-k}(\mathfrak g,N\otimes_R M)$$

in the (co)homology of $\mathfrak g$ which, in turn, is induced by the standard Hopf algebra structure on the universal algebra $U\mathfrak g$ of $\mathfrak g$ obtained when the elements of $\mathfrak g$ are required to be primitive. (When $\mathfrak g$ is not projective, these statements are true for the corresponding relative homological algebra.) While this Hopf algebra structure is lurking behind, there is no need for us to make it explicit, and we refrain from spelling out details. Our crucial observation is that, though in general $U(A,L)$ no longer inherits a Hopf algebra structure, the relevant cap pairing is still available.

Proof. At first, we view L as a Lie algebra over R and write $K_R L$ for its Koszul complex in the category of R-modules. Consider the cap pairing

$$\cap\colon (N \otimes_{UL} K_R L) \otimes_R \mathrm{Alt}_R(L, M) \longrightarrow (N \otimes_R M) \otimes_{UL} K_R L \tag{3.3.3}$$

of chain complexes in the category of R-modules. As observed above, this pairing is induced by the shuffle diagonal map (3.1.1) on the exterior R-algebra $\Lambda_R L$ and hence may be written

$$\cap\colon (N \otimes_R \Lambda_R L) \otimes_R \mathrm{Hom}_R(\Lambda_R L, M) \longrightarrow (N \otimes_R M) \otimes_R \Lambda_R L. \tag{3.3.4}$$

Likewise, taking now into account the A-module structure on L and using the shuffle diagonal map (3.1.4) on the exterior A-algebra $\Lambda_A L$, we obtain a similar pairing

$$\cap\colon (N \otimes_A \Lambda_A L) \otimes_R \mathrm{Hom}_A(\Lambda_A L, M) \longrightarrow (N \otimes_A M) \otimes_A \Lambda_A L \tag{3.3.5}$$

of graded R-modules. Replacing the tensor product "$\otimes_R$" with the tensor product "$\otimes_A$" over A we would even obtain a pairing of graded A-modules but below we shall stick to the pairing (3.3.5) of graded R-modules. (We remarked earlier that the distinction between graded A-objects and differential graded R-modules due to the in general non-trivial action of L on A persists throughout.) In view of the isomorphisms

$$\mathrm{Hom}_A(\Lambda_A L, M) \cong \mathrm{Alt}_A(L, M) \cong \mathrm{Hom}_{U(A,L)}(K(A,L), M) \tag{3.3.6}$$

and

$$\begin{gathered} N \otimes_A \Lambda_A L \cong N \otimes_{U(A,L)} K(A,L) \\ (N \otimes_A M) \otimes_A \Lambda_A L \cong (N \otimes_A M) \otimes_{U(A,L)} K(A,L) \end{gathered} \tag{3.3.7}$$

of graded A-modules, differentials being ignored, the pairing (3.3.5) has the form

$$\begin{aligned} \cap\colon (N \otimes_{U(A,L)} K(A,L)) \otimes_R \mathrm{Hom}_{U(A,L)}(K(A,L), M)& \\ \longrightarrow (N \otimes_A M) \otimes_{U(A,L)} K(A,L).& \end{aligned} \tag{3.3.8}$$

We now assert that this pairing is in fact compatible with the differentials. In order to see this we recall that (i) the inclusion

$$\mathrm{Alt}_A(L, M) \subseteq \mathrm{Alt}_R(L, M)$$

is a morphism of chain complexes, and that (ii) the projection maps

$$N \otimes_R \Lambda_R L \longrightarrow N \otimes_A \Lambda_A L$$

and

$$(N \otimes_R M) \otimes_R \Lambda_R L \longrightarrow (N \otimes_A M) \otimes_A \Lambda_A L$$

are morphisms of chain complexes, too. These two observations are due to Rinehart [21]. Hence restricting to $\mathrm{Alt}_A(L, M) = \mathrm{Hom}_A(\Lambda_A L, M)$, from (3.3.4), we obtain the pairing

$$(N \otimes_R \Lambda_R L) \otimes_R \mathrm{Hom}_A(\Lambda_A L, M) \longrightarrow (N \otimes_R M) \otimes_R \Lambda_R \tag{3.3.9}$$

of chain complexes. The latter, in turn, fits into the commutative diagram

$$\begin{array}{ccc} (N \otimes_R \Lambda_R L) \otimes_R \mathrm{Hom}_A(\Lambda_A L, M) & \longrightarrow & (N \otimes_R M) \otimes_R \Lambda_R L \\ \big\downarrow & & \big\downarrow \\ (N \otimes_A \Lambda_A L) \otimes_R \mathrm{Hom}_A(\Lambda_A L, M) & \longrightarrow & (N \otimes_A M) \otimes_A \Lambda_A L \end{array} \tag{3.3.10}$$

having the upper row and the two vertical maps morphisms of chain complexes. Since the vertical maps are surjective, this implies that the lower row is also a morphism of chain complexes. The latter induces the cap pairing

$$\cap\colon \mathrm{H}_\ell(L, N) \otimes_R \mathrm{H}^k(L, M) \to \mathrm{H}_{\ell-k}(L, N \otimes_A M)$$

which we are looking for. □

Suppose now that L is a duality (R, A)-Lie algebra of rank n. Let $C = \Lambda_A^n L^*$, the dualizing module for L. The duality isomorphism (2.11.2), for $N = C$ and $k = n$, takes the form

$$\psi\colon \mathrm{H}_n(L, C) \to \mathrm{Hom}_U(C, C). \tag{3.4}$$

Let $e \in \mathrm{H}_n(L, C)$ be the class which under this isomorphism goes to Id_C. We refer to it as the *fundamental class* of the Lie-Rinehart algebra (A, L). We note that $\mathrm{Hom}_U(C, C) \cong \mathrm{H}^0(L, A)$. When A and L are the algebra of smooth functions and Lie algebra of smooth vector fields on a smooth (connected) n-dimensional manifold W, $\mathrm{H}^0(L, A) \cong \mathrm{H}^0(W, \mathbb{R}) \cong \mathbb{R}$ (the constant functions) and, for W compact, Poincaré duality identifies $\mathrm{H}^0(W, \mathbb{R})$ with $\mathrm{H}_n(W, \mathbb{R}_t)$ where $\mathbb{R}_t$ refers to the real valued local system on W arising from the orientation bundle which is non-trivial if and only if W is not orientable; our fundamental class e can then be identified with the ordinary fundamental class of W. See Remark 4.11 below for details.

For a general Lie-Rinehart algebra (A, L), with $N = C$, the cap pairing (3.3.1) yields

$$\cap\colon \mathrm{H}_n(L, C) \otimes_R \mathrm{H}^k(L, M) \to \mathrm{H}_{n-k}(L, C \otimes_A M) \tag{3.5}$$

while with $M = \mathrm{Hom}_A(C, N)$ and the ordinary evaluation pairing $\mathrm{Hom}_A(C, N) \otimes_A C \to N$, we get

$$\cap\colon \mathrm{H}_n(L, C) \otimes_R \mathrm{H}^k(L, \mathrm{Hom}_A(C, N)) \to \mathrm{H}_{n-k}(L, N). \tag{3.6}$$

Theorem 3.7. *Given a duality (R, A)-Lie algebra L of rank n, cap-product with its fundamental class e produces isomorphisms*

$$(e \cap \cdot\,)\colon \mathrm{H}^k(L, M) \to \mathrm{H}_{n-k}(L, C \otimes_A M) \tag{3.7.1}$$

for all non-negative integers k and all left (A, L)-modules M and, furthermore, isomorphisms

$$(e \cap \cdot\,)\colon \mathrm{H}^{n-k}(L, \mathrm{Hom}_A(C, N)) \to \mathrm{H}_k(L, N) \tag{3.7.2}$$

for all non-negative integers k and all right (A,L)-modules N.

Consequently the duality isomorphisms may be taken to be natural in any reasonable sense. The proof to be given below owes much to the corresponding proof of Theorem 9.5 in [2] but the proof in [2] cannot just be adapted to Lie-Rinehart algebras; the basic difference is that, for a Lie-Rinehart algebra (A,L), the algebra $U(A,L)$ acts non-trivially on A while in [2] (as always in group cohomology) the action of the group ring on the ground ring is trivial.

Proof. It proceeds in five steps.

Step 1. Here we prove that, in the top dimension n, the duality isomorphism

$$\psi\colon \mathrm{H}_n(L,N) \longrightarrow \mathrm{Hom}_U(C,N) = \mathrm{H}^0(L, \mathrm{Hom}_A(C,N)) \tag{3.7.3}$$

is given by the formula

$$(\psi(u))(c) = u \cap c, \quad u \in \mathrm{H}_n(L,N),\ c \in C = \mathrm{H}^n(L,U). \tag{3.7.4}$$

Indeed, for the highest non-zero A-exterior power $\Lambda_A^n L$ of L, the pairing (3.3.5) comes down to the evaluation pairing

$$(N \otimes_A \Lambda_A^n L) \otimes_R \mathrm{Hom}_A(\Lambda_A^n L, M) \longrightarrow N \otimes_A M \tag{3.7.5}$$

which, on homology, induces the pairing

$$\cap\colon \mathrm{H}_n(L,N) \otimes_R \mathrm{H}^n(L,M) \longrightarrow \mathrm{H}_0(L, N \otimes_A M) = (N \otimes_A M) \otimes_U A.$$

In particular, with $M = U$, U being viewed as a left U-module, we obtain the pairing

$$\cap\colon \mathrm{H}_n(L,N) \otimes_R \mathrm{H}^n(L,U) \longrightarrow (N \otimes_A U) \otimes_U A \cong N$$

which, with reference to the right U-module structure on $\mathrm{H}^n(L,U)$ coming from the right U-module structure on itself (which does not come into play in the construction of $\mathrm{H}^n(L,U)$) and that on N, is compatible with the right U-module structures. The adjoint of this pairing is the morphism of R-modules

$$\mathrm{H}_n(L,N) \longrightarrow \mathrm{Hom}_U(\mathrm{H}^n(L,U), N), \quad u \mapsto (c \mapsto u \cap c).$$

The latter, in turn, is plainly induced by the adjoint

$$N \otimes_A \Lambda_A^n L \longrightarrow \mathrm{Hom}_U(\mathrm{Hom}_A(\Lambda_A^n L, U), N) \tag{3.7.6}$$

of (3.7.5) with $M = U$. However, when the resolution K in (1.4.2) is taken to be the Rinehart complex (2.9.2), which under these circumstances is a projective resolution of A in the category of left $U(A,L)$-modules, (3.7.6) is just a rewrite of the isomorphism (1.4.2) in the top dimension. Since (1.4.2) induces (3.7.3), where $C = \mathrm{H}^n(L,U)$, it follows that (3.7.3) is given by (3.7.4) as asserted. In particular, since $\psi(e) = \mathrm{Id}_C$, taking $N = C$, we obtain

$$e \cap c = c, \quad c \in C = \mathrm{H}^n(L,U). \tag{3.7.7}$$

Step 2. Now we claim that, still in the top dimension, the duality isomorphism

$$\phi\colon C\otimes_U M \longrightarrow \mathrm{H}^n(L,M) \tag{3.7.8}$$

is given by the following formula: Let $\zeta\colon K_n \to U$ represent $c \in C = \mathrm{H}^n(L,U)$, let $x \in M$, let $\omega_x\colon U \to M$ be given by $\omega_x(1) = x$, and write $(\omega_x)_*\colon \mathrm{H}^n(L,U) \to \mathrm{H}^n(L,M)$ for the induced map; then

$$\phi(c\otimes x) = (\omega_x)_* c. \tag{3.7.9}$$

In fact, the duality isomorphism (3.7.8) is induced by the canonical isomorphism Φ from $K^*\otimes_U M$ onto $\mathrm{Hom}_U(K,M)$ where K still refers to the Rinehart complex (2.9.2), cf. (1.4.1). However, under this isomorphism, $\zeta\otimes x$ goes to the composite of ζ with ω_x whence (3.7.9) holds.

Step 3. Now we claim that, still in the top dimension, the special case

$$(e\cap\cdot)\colon \mathrm{H}^n(L,M) \longrightarrow \mathrm{H}_0(L, C\otimes_A M) = C\otimes_U M \tag{3.7.10}$$

of (3.7.1) is the inverse of (3.7.8) and hence, in particular, is an isomorphism. In order to see this, as before, let $x \in M$ and let $\omega_x\colon U \to M$ be given by $\omega_x(1) = x$. Consider the diagram

$$\begin{tikzcd} C\otimes_U U \arrow[r, "\mathrm{Id}"] \arrow[d, "(\omega_x)_*"] & \mathrm{H}^n(L,U) \arrow[r, "(e\cap\cdot)"] \arrow[d, "(\omega_x)_*"] & C\otimes_U U \arrow[d, "(\omega_x)_*"] \\ C\otimes_U M \arrow[r, "\phi"] & \mathrm{H}^n(L,M) \arrow[r, "(e\cap\cdot)"] & C\otimes_U M \end{tikzcd} \tag{3.7.11}$$

By virtue of what has been proved in Step 2, this diagram is commutative and, in view of (3.7.7), the composite of the top row is the identity map. Since $x \in M$ was arbitrary, this shows that the composite of the bottom row is the identity of $C\otimes_U M$, hence $(e\cap\cdot) = \phi^{-1}$ is an isomorphism.

Step 4. Here we show that (3.7.1) is an isomorphism for every non-negative integer k. In order to see this, let $0 \to M_1 \to P \to M \to 0$ be a short exact sequence of left U-modules with P projective. Since C is projective as an A-module, the resulting sequence

$$0 \to C\otimes_A M_1 \to C\otimes_A P \to C\otimes_A M \to 0$$

of right U-modules, with right U-module structures given by (2.4), is still exact, and hence naturality of the cap product yields the commutative diagram

$$\begin{tikzcd} \mathrm{H}^{n-1}(L,P) \arrow[r] \arrow[d] & \mathrm{H}^{n-1}(L,M) \arrow[r] \arrow[d] & \mathrm{H}^n(L,M_1) \arrow[r] \arrow[d, "\cong"] & \mathrm{H}^n(L,P) \arrow[d, "\cong"] \\ 0 \arrow[r] & \mathrm{H}_1(L,C\otimes_A M) \arrow[r] & \mathrm{H}_0(L,C\otimes_A M_1) \arrow[r] & \mathrm{H}_0(L,C\otimes_A P), \end{tikzcd}$$

each vertical morphism being given by $(e\cap\cdot)$. However, from (2.11.1), with $M = P$, we deduce

$$\mathrm{H}^{n-1}(L,P) \cong \mathrm{H}_1(L,C\otimes_A P) \cong \mathrm{Tor}_U(C,P) = 0$$

since P is a projective U-module. Hence

$$(e\cap\cdot)\colon \mathrm{H}^{n-1}(L,M) \to \mathrm{H}_1(L, C\otimes_A M)$$

is an isomorphism, too, and the argument can be iterated. This establishes the isomorphisms (3.7.1).

Step 5. The isomorphisms (3.7.2) are now a formal consequence of those already established. Indeed, taking $M = \mathrm{Hom}_A(C,N)$, we get

$$(e\cap\cdot)\colon \mathrm{H}^k(L,\mathrm{Hom}_A(C,N)) \to \mathrm{H}_{n-k}(L, C\otimes_A \mathrm{Hom}_A(C,N)).$$

However, $C\otimes_A \mathrm{Hom}_A(C,N) \cong N$ as right (A,L)-modules. This establishes the isomorphisms (3.7.2) as well. $\square$

Let L be a duality (R,A)-Lie algebra of rank n. Its duality properties give rise to certain bilinear pairings in its cohomology, in the following way: Let $M_1\otimes_A M_2 \to M$ be a pairing of left (A,L)-modules. The naturality of the cap pairings and the compatibility properties between the cup- and cap pairings imply that the requisite duality isomorphisms combine to a commutative diagram

$$\begin{CD}
\mathrm{H}^k(L,M_1)\otimes_R \mathrm{H}^{n-k}(L,M_2) @>>> \mathrm{H}^n(L,M)\\
@V{(e\cap\cdot)\otimes_R \mathrm{Id}}VV @VV{(e\cap\cdot)}V\\
\mathrm{H}_{n-k}(L,C\otimes_A M_1)\otimes_R \mathrm{H}^{n-k}(L,M_2) @>>> C\otimes_U M
\end{CD}\tag{3.8}$$

having as upper row the corresponding cup pairing (3.2), as lower row the cap pairing (3.3.1) (with the notation k, ℓ, M, N adjusted appropriately) and, furthermore, all vertical maps isomorphisms; here $U = U(A,L)$. From (3.8) we obtain the bilinear pairing

$$\mathrm{H}^k(L,M_1)\otimes_R \mathrm{H}^{n-k}(L,M_2) \to C\otimes_U M. \tag{3.9}$$

The question we will study in the next section is whether, for suitable pairings $M_1\otimes_A M_2 \to M$, nondegenerate in a suitable way, the pairing (3.9) may be nondegenerate, in an appropriate sense.

Remark 3.10. This question will, in general, not have a naive solution, as the following example shows: Under the circumstances of (2.10), suppose that the ground ring is that of the reals, $\mathbb{R}$, let A be the algebra of smooth real functions on a smooth real manifold W and $\mathfrak{g}$ a real Lie algebra acting infinitesimally on W, and let $L = A\otimes_{\mathbb{R}}\mathfrak{g}$, the $(\mathbb{R},A)$-Lie algebra introduced in (2.10). Further, let $\mathcal{O}$ be the space of sections of the orientation bundle of W, compactly supported when W is not compact, with its canonical left (A,L)-module structure, and denote by ω_A the dualizing module of $\mathrm{Der}(A) = \mathrm{Vect}(W)$; under the present circumstances, ω_A is just the highest non-zero exterior power of the space of sections of the cotangent bundle of W. Real valued bilinear cohomology pairings can then be obtained by integration over W, in the following way: Consider a bilinear pairing

$$M_1\otimes_A M_2 \to \mathrm{Hom}_A(C_L, \omega_A\otimes_A \mathcal{O})$$

of left (A,L)-modules arising from a nondegenerate pairing of smooth vector bundles on W, the target $\mathrm{Hom}_A(C_L, \omega_A \otimes_A \mathcal{O})$ being endowed with the obvious left (A,L)-module structure (2.2). The resulting pairing (3.9) then has its values in the space $\omega_A \otimes_{U(A,L)} \mathcal{O}$ which amounts to that of (compactly supported) densities on W, with the $\mathfrak{g}$-action being divided out. Integration yields a map from this space to the reals but this map will not be an isomorphism unless the structure map from L to $\mathrm{Vect}(W)$ is surjective whence the resulting real-valued pairing may be nondegenerate only if the space of densities with the $\mathfrak{g}$-action being divided out amounts to a single copy of the reals. This confirms and explains the empirical observations made in [10]. In particular, the example given there shows that $\omega_A \otimes_{U(A,L)} \mathcal{O}$ may well be larger than a single copy of the reals. The cure is provided by a notion of nondegeneracy over more general rings: When the structure map from L to $\mathrm{Vect}(W)$ is not surjective, in a sense, the resulting pairing (3.9) looks nondegenerate over $\omega_A \otimes_{U(A,L)} \mathcal{O}$ rather than over $\mathbb{R}$ except that this does not make sense as it stands since $\omega_A \otimes_{U(A,L)} \mathcal{O}$ is not a ring. By means of an additional piece of structure which may or may not exist, that is to say, by means of the notion of a *trace*, we shall make precise this idea of nondegeneracy over more general rings in the next section.

4. Poincaré duality

In this section we shall show that, for certain Lie-Rinehart algebras, Poincaré duality holds in their cohomology, the terminology "Poincaré" referring to the nondegeneracy of certain pairings of the kind (3.9) in a sense which we are about to make precise. As a special case, we obtain a new proof of Poincaré duality in the de Rham cohomology of a smooth manifold. We shall use the descriptions of the duality isomorphisms in terms of the cap product with the fundamental class given in (3.7). This entails all the requisite naturality properties.

As before, let A be a commutative algebra and L a duality (R,A)-Lie algebra of rank n. A left (A,L)-module M which, as an A-module, is finitely generated and projective, will henceforth be referred to as a left *duality* (A,L)-module; likewise a right (A,L)-module N which, as an A-module, is finitely generated and projective, will be called a right *duality* (A,L)-module. We shall say that a *pretrace* for L consists of a left (A,L)-module $\mathcal{O}$, referred to as its *trace module*, together with an isomorphism

$$t\colon \mathrm{H}^n(L,\mathcal{O}) \to R \tag{4.1}$$

of R-modules; a pretrace $(\mathcal{O},t)$ for L will be said to be a *trace* for L provided a *weak* form of *Poincaré duality* holds in the sense that, for every left duality (A,L)-module M, $\mathrm{H}^0(L,M) = \mathrm{Hom}_U(A,M)$ and $\mathrm{H}^n(L,\mathrm{Hom}_A(M,\mathcal{O}))$ are dually paired, that is to say, the canonical morphism

$$\begin{aligned} \mathrm{Hom}_U(A,M) &\to \mathrm{Hom}_R(\mathrm{H}^n(L,\mathrm{Hom}_A(M,\mathcal{O})),\mathrm{H}^n(L,\mathcal{O})) \\ &\xrightarrow{t_*} \mathrm{Hom}_R(\mathrm{H}^n(L,\mathrm{Hom}_A(M,\mathcal{O})),R) \end{aligned} \tag{4.2.1}$$

of R-modules is an isomorphism, the morphism t_* being the isomorphism induced by the trace.

A variant of this notion of trace is obtained when the canonical morphism (4.2.1) is only required to be an isomorphism for every left duality (A, L)-module M belonging to a suitable class of left duality (A, L)-modules. An example will be given in (4.15) below.

Our notion of 'trace' is similar to a corresponding one in the theory of Serre duality for the cohomology of projective sheaves on a projective scheme, cf. e. g. [14], but this terminology does not necessarily coincide with other usages thereof in de Rham and Poisson cohomology.

Let C_L denote the dualizing module of L, cf. (2.10). In view of the naturality of the duality isomorphism $e \cap \cdot : \mathrm{H}^n(L, \mathcal{O}) \to \mathrm{C}_L \otimes_U \mathcal{O}$, cf. the commutative diagram (3.8), a trace may as well be described as an isomorphism

$$(4.3) \qquad t \colon C_L \otimes_U \mathcal{O} \to R$$

of R-modules such that, for every left duality (A, L)-module M, the canonical morphism

$$(4.4.1) \quad \mathrm{Hom}_U(A, M) \to \mathrm{Hom}_R(C_L \otimes_U \mathrm{Hom}_A(M, \mathcal{O}), C_L \otimes_U \mathcal{O}) \xrightarrow{t_*} \mathrm{Hom}_R(C_L \otimes_U M, R)$$

is an isomorphism of R-modules.

Example 4.5. Let A be a commutative algebra and $\mathfrak{g}$ an ordinary Lie algebra over A which we suppose finitely generated and projective as an A-module, of constant rank k. To subsume this example under our general theory, we take A as the ground ring, written R, and distinguish A and R deliberately in notation although A and R coincide, but this notational distinction will enable us to abstract from the special situation of this example. We then view $(A, \mathfrak{g})$ as a duality (R, A)-Lie algebra of rank k. A typical case arises from a principal G-bundle $\xi \colon P \to B$ with structure group G: its Lie algebra $\mathfrak{g}(\xi)$ of infinitesimal gauge transformations is the space of sections of the adjoint bundle and inherits a Lie algebra structure over the ring of smooth functions on B. Returning to a general Lie algebra $\mathfrak{g}$ over A of constant rank k, let $\mathcal{O}_{\mathfrak{g}} = \Lambda^{\mathrm{k}} \mathfrak{g}$, the top exterior power of $\mathfrak{g}$, with its standard left $\mathfrak{g}$-module structure. With reference to the canonical right $\mathfrak{g}$-module structure on $\mathrm{Hom}_A(U\mathfrak{g}, A)$ induced by the left $\mathfrak{g}$-structure on $U\mathfrak{g}$, we have the homology group

$$(4.5.1) \qquad \mathrm{H}_k(\mathfrak{g}, \mathrm{Hom}_A(U\mathfrak{g}, A));$$

from the right $\mathfrak{g}$-structure on $U\mathfrak{g}$ which remains free under the construction of $\mathrm{H}_k(\mathfrak{g}, \mathrm{Hom}_A(U\mathfrak{g}, A))$, this homology group inherits a left $\mathfrak{g}$-module structure and, with this structure, $\mathrm{H}_k(\mathfrak{g}, \mathrm{Hom}_A(U\mathfrak{g}, A))$ is canonically isomorphic to $\mathcal{O}_{\mathfrak{g}}$ as a left $\mathfrak{g}$-module; in fact, $\mathrm{H}_k(\mathfrak{g}, \mathrm{Hom}_A(U\mathfrak{g}, A))$ and $\mathrm{H}^0(\mathfrak{g}, \mathrm{Hom}_A(C_{\mathfrak{g}}, \mathrm{Hom}_A(U\mathfrak{g}, A)))$ are canonically isomorphic by duality, and

$$\mathrm{H}^0(\mathfrak{g}, \mathrm{Hom}_A(C_{\mathfrak{g}}, \mathrm{Hom}_A(U\mathfrak{g}, A))) \cong \mathrm{Hom}_A(C_{\mathfrak{g}}, A) \cong \mathcal{O}_{\mathfrak{g}}.$$

As a right (A, L)-module, the dualizing module $C_{\mathfrak{g}}$ is therefore naturally isomorphic to $\mathrm{Hom}_A(\mathcal{O}_{\mathfrak{g}}, A) = \mathrm{Hom}_A(\Lambda^k \mathfrak{g}, A)$, the right $\mathfrak{g}$-module structure on the latter being induced by the left $\mathfrak{g}$-module structure on $\Lambda^k \mathfrak{g}$. Then

$$(4.5.2) \qquad C_{\mathfrak{g}} \otimes_A \mathcal{O}_{\mathfrak{g}} \cong \mathrm{Hom}_A(C_{\mathfrak{g}}, C_{\mathfrak{g}}) \cong A$$

inherits a right $\mathfrak{g}$-module structure which has to be trivial, since manifestly

$$\mathrm{Hom}_{\mathfrak{g}}(C_{\mathfrak{g}}, C_{\mathfrak{g}}) \cong R = A.$$

Consequently, in the top dimension, the duality isomorphism for the left $\mathfrak{g}$-module $\mathcal{O}_{\mathfrak{g}}$ takes the form

$$\mathrm{H}^k(\mathfrak{g}, \mathcal{O}_{\mathfrak{g}}) \xrightarrow{e\cap\,\cdot} \mathrm{H}_0(\mathfrak{g}, A) \cong A$$

and this in fact defines a trace

$$t\colon \mathrm{H}^k(\mathfrak{g}, \mathcal{O}_{\mathfrak{g}}) \longrightarrow A. \tag{4.5.3}$$

It is readily seen that the property (4.2.1) holds; we do not give the details. Thus an ordinary Lie algebra $\mathfrak{g}$ of finite constant rank always comes with a trace, having as trace module the highest non-zero homology group $\mathrm{H}_{\mathrm{top}}(\mathfrak{g}, \mathrm{Hom}_A(U\mathfrak{g}, A))$, and this homology group is canonically isomorphic to the highest non-zero exterior power $\Lambda^{\mathrm{top}}\mathfrak{g}$ of $\mathfrak{g}$; here A is to be viewed as a right $\mathfrak{g}$-module with trivial $\mathfrak{g}$-action. Moreover, given a left $\mathfrak{g}$-module M, the canonical (or cap) pairing

$$\mathrm{H}^\ell(\mathfrak{g}, M) \otimes_A \mathrm{H}_\ell(\mathfrak{g}, \mathrm{Hom}_A(M, A)) \longrightarrow \mathrm{H}_0(\mathfrak{g}, A) \xrightarrow{\cong} A \tag{4.5.4}$$

is manifestly nondegenerate because it arises from a nondegenerate pairing of A-chain complexes. In view of the obvious isomorphisms

$$C_{\mathfrak{g}} \otimes_A \mathrm{Hom}_A(M, \mathcal{O}_{\mathfrak{g}}) \cong \mathrm{Hom}_A(M, C_{\mathfrak{g}} \otimes_A \mathcal{O}_{\mathfrak{g}}) \cong \mathrm{Hom}_A(M, A),$$

the nondegeneracy of (4.5.4) implies at once that of the cup pairing

$$\mathrm{H}^\ell(\mathfrak{g}, M) \otimes_A \mathrm{H}^{k-\ell}(\mathfrak{g}, \mathrm{Hom}_A(M, \mathcal{O}_{\mathfrak{g}})) \longrightarrow \mathrm{H}^k(\mathfrak{g}, \mathcal{O}_{\mathfrak{g}}) \xrightarrow{t} A. \tag{4.5.5}$$

In particular, the canonical duality pairing between $C_{\mathfrak{g}}$ and $\mathcal{O}_{\mathfrak{g}}$ may be described as the canonical evaluation pairing

$$\mathrm{H}^k(\mathfrak{g}, U\mathfrak{g}) \otimes_A \mathrm{H}_k(\mathfrak{g}, \mathrm{Hom}_A(U\mathfrak{g}, A)) \longrightarrow \mathrm{H}_0(\mathfrak{g}, A) \cong A. \tag{4.5.6}$$

The pairing (4.5.5) is the Poincaré duality pairing in ordinary Lie algebra cohomology, phrased here over an arbitrary commutative algebra A. The corresponding Lie-Rinehart structure is trivial and the duality properties are well known; we therefore refer to this pairing as the *classical* duality pairing. For example, given a principal bundle ξ over a smooth manifold B, the pairing (4.5.5) is available over the ring A of smooth functions on B for the Lie algebra $\mathfrak{g}(\xi)$ of infinitesimal gauge transformations.

For a general duality Lie-Rinehart algebra (A, L), the highest non-zero exterior power of L over A with its naive left L-module structure does *not* yield a left (A, L)-module and hence cannot naively serve for the construction of a trace module, and a trace is an additional piece of structure. We now give a different description of a trace involving the appropriate direct generalization of the trivial right $\mathfrak{g}$-module structure on the copy of A occurring in the top homology of $\mathfrak{g}$ with coefficients in $\mathrm{Hom}_A(U\mathfrak{g}, A)$ just explained, cf. (4.5.1). This description will be used in the next section to construct traces for extensions of Lie-Rinehart algebras.

Proposition 4.6. *Let (A,L) be a duality Lie-Rinehart algebra.*
(i) *Given a trace $(\mathcal O,t)$ for (A,L), the projective rank one A-module $V = C_L \otimes_A \mathcal O$ with right (A,L)-module structure* (2.4) *has the property that t induces an isomorphism $\iota\colon V\otimes_U A \to R$ and that, for every right duality (A,L)-module N, the canonical map*

$$\mathrm{Hom}_U(N,V) \longrightarrow \mathrm{Hom}_R(N\otimes_U A, V\otimes_U A) \xrightarrow{\iota_*} \mathrm{Hom}_R(\mathrm H_0(L,N),R) \tag{4.6.1}$$

is an isomorphism.
(ii) *Conversely, given a right (A,L)-module V which, as an A-module, is projective of rank one, together with an isomorphism $\iota\colon V\otimes_U A\to R$, let $\mathcal O = \mathrm{Hom}_A(C_L,V)$, with its induced left (A,L)-module structure* (2.3) *so that $C_L\otimes_A\mathcal O$ is canonically isomorphic to V; then the composite*

$$\mathrm H^n(L,\mathcal O) \longrightarrow \mathrm H_0(L, C_L\otimes_A \mathcal O) \cong \mathrm H_0(L,V) \cong V\otimes_U A \xrightarrow{\iota} R \tag{4.6.2}$$

yields a trace provided that, for every right duality (A,L)-module N, the canonical map

$$\mathrm{Hom}_U(N,V) \longrightarrow \mathrm{Hom}_R(N\otimes_U A, V\otimes_U A) \xrightarrow{\iota_*} \mathrm{Hom}_R(\mathrm H_0(L,N),R) \tag{4.6.3}$$

is an isomorphism.

Proof. The argument for (i) is straightforward. We briefly indicate one for (ii): Given a left duality (A,L)-module M, let $N=\mathrm{Hom}_A(M,V)$, with right (A,L)-module structure (2.4). Then

$$\mathrm{Hom}_U(A,M)\cong \mathrm{Hom}_U(\mathrm{Hom}_A(M,V),V)\cong \mathrm{Hom}_U(N,V)$$

and $\mathrm{Hom}_A(M,\mathcal O)\cong \mathrm{Hom}_A(C_L,N)$ whence

$$\mathrm{Hom}_R(\mathrm H^n(L,\mathrm{Hom}_A(M,\mathcal O)),R)\cong \mathrm{Hom}_R(\mathrm H^n(L,\mathrm{Hom}_A(C_L,N)),R)$$

and, by duality,

$$\mathrm{Hom}_R(\mathrm H^n(L,\mathrm{Hom}_A(C_L,N)),R)\cong \mathrm{Hom}_R(\mathrm H_0(L,N),R).$$

Since (4.6.3) is an isomorphism, (4.2.1) is an isomorphism, too. □

The right (A,L)-module V in (4.6) arises by abstraction from the isomorphism (4.5.2). In all the examples that we know, as an A-module, this module is actually free of rank 1, that is, just a copy of A, so that the two modules C_L and $\mathcal O$ are dual to each other as A-modules.

Example 4.7. Let the ground ring be that of the reals, $\mathbb R$, let W be a smooth real n-dimensional manifold W, A its algebra of smooth functions, L the $(\mathbb R,A)$-Lie algebra $\mathrm{Vect}(W)$ of smooth vector fields on W, and let $\mathcal O$ be the space of compactly supported sections of the orientation bundle of W, with its canonical flat connection and hence left (A,L)-module structure. The dualizing module C_L of L is just the space ω_A of smooth n-forms on W, with right (A,L)-module structure given by (2.9.1), for $M=A$. Then the tensor product $\omega_A\otimes_A\mathcal O$ is the space of compactly

supported densities on W, and the integration map from $\omega_A \otimes_A \mathcal{O}$ to $\mathbb{R}$ induces a trace

$$t\colon \omega_A \otimes_U \mathcal{O} \to \mathbb{R} \tag{4.7.1}$$

for L. In fact, a left duality (A, L)-module M arises as the space of sections of a flat vector bundle ζ_M on W, and (4.2.1) comes down to the fact that $\mathrm{H}^0(W, \zeta_M)$ and $\mathrm{H}^n_{\mathrm{cs}}(W, \zeta^*_M)$ are dually paired where H_{cs} refers to cohomology with compact supports and ζ^*_M to the dual bundle. Notice that when W is compact $\mathcal{O}$ is projective as an A-module and there is no need to talk about compactly supported forms; furthermore, in this case, $\omega_A \otimes_A \mathcal{O}$ is a free A-module of dimension one — that of densities on W — and thus, endowed with the right (A, L)-module structure (2.4), A carries a right (A, L)-module structure; when A is viewed as a right (A, L)-module in this way, we denote it henceforth by A_r. Integration then induces an isomorphism $\iota\colon A_r \otimes_U A \to \mathbb{R}$ in such a way that the right (A, L)-module V in (4.6) is just A_r and that ι satisfies the conditions of (4.6)(ii). Thus, with t being defined by (4.6.2), $(\mathcal{O}, t)$ is a trace for (A, L).

We now return to a general duality (R, A)-Lie algebra L, for an arbitrary R-algebra A. Given a trace $(\mathcal{O}, t)$ for L having $\mathcal{O}$ projective as an A-module, in (4.2.1) we can replace M with $\mathrm{Hom}_A(M, \mathcal{O})$; thus instead of (4.2.1) we can then equivalently define a trace by the more appealing requirement that the canonical morphism

$$\mathrm{Hom}_U(M, \mathcal{O}) \to \mathrm{Hom}_R(\mathrm{H}^n(L, M), \mathrm{H}^n(L, \mathcal{O})) \xrightarrow{t_*} \mathrm{Hom}_R(\mathrm{H}^n(L, M), R) \tag{4.2.2}$$

be an isomorphism of R-modules for *every* left duality (A, L)-module M; likewise, (4.4.1) to be an isomorphism is then equivalent to the canonical morphism

$$\mathrm{Hom}_U(M, \mathcal{O}) \to \mathrm{Hom}_R(C_L \otimes_U M, C_L \otimes_U \mathcal{O}) \xrightarrow{t_*} \mathrm{Hom}_R(C_L \otimes_U M, R) \tag{4.4.2}$$

of R-modules being an isomorphism. Technically the definition given at the beginning of the present section allows for more freedom, though, for example when Lie-Rinehart algebras defined over non-compact smooth manifolds come into play; see e. g. (4.10) below. The usual abstract nonsense establishes the following the proof of which is left to the reader.

Lemma 4.8. *A trace for L having trace module $\mathcal{O}$ projective as an A-module is unique up to isomorphism. More precisely, given two such traces $(\mathcal{O}_1, t_1)$ and $(\mathcal{O}_2, t_2)$, there is a unique isomorphism of left (A, L)-modules between $\mathcal{O}_1$ and $\mathcal{O}_2$ identifying the two traces.* □

Let L be a duality (R, A)-Lie algebra, and suppose that L is endowed with a trace $(\mathcal{O}, t)$. Given a left (A, L)-module M, in view of what has been said in Section 3 before, the canonical pairing $M \otimes_A \mathrm{Hom}_A(M, \mathcal{O}) \to \mathcal{O}$ of A-modules induces the cap pairing

$$\cap\colon \mathrm{Tor}^U_k(C_L, M) \otimes_R \mathrm{Ext}^k_U(M, \mathcal{O}) \to C_L \otimes_U \mathcal{O} \xrightarrow{\iota} R \tag{4.9.1}$$

and the cup pairing

$$\cup\colon \mathrm{H}^k(L, M) \otimes_R \mathrm{H}^{n-k}(L, \mathrm{Hom}_A(M, \mathcal{O})) \to \mathrm{H}^n(L, \mathcal{O}) \xrightarrow{t} R. \tag{4.9.2}$$

The cap pairing (4.9.1) may manifestly be written

$$\cap\colon \mathrm{H}_k(L, C_L \otimes_A M) \otimes_R \mathrm{H}^k(L, \mathrm{Hom}_A(M, \mathcal{O})) \longrightarrow C_L \otimes_U \mathcal{O} \xrightarrow{\iota} R \tag{4.9.3}$$

as well.

Likewise, given a right (A, L)-module N, taking $M = \mathrm{Hom}_A(N, V)$ with left (A, L)-module structure (2.3), where $V = C_L \otimes_A \mathcal{O}$, cf. (4.6), with right (A, L)-module structure (2.4), we obtain the corresponding cap pairing (3.3.1) which now has the form

$$\cap\colon \mathrm{H}_k(L, N) \otimes_R \mathrm{H}^k(L, \mathrm{Hom}_A(N, V)) \longrightarrow \mathrm{H}_0(L, N \otimes_A \mathrm{Hom}_A(N, V));$$

combining it with the morphism on H_0 induced by the canonical morphism

$$N \otimes_A \mathrm{Hom}_A(N, V) \longrightarrow V$$

of right (A, L)-modules, we obtain the pairing

$$\cap\colon \mathrm{H}_k(L, N) \otimes_R \mathrm{H}^k(L, \mathrm{Hom}_A(N, V)) \longrightarrow \mathrm{H}_0(L, V) \xrightarrow{\iota} R \tag{4.9.4}$$

which we refer to as *cap pairing* as well. Given a left (A, L)-module M, the pairing (4.9.3) coincides with the pairing (4.9.4) for the right (A, L)-module $N = C_L \otimes_A M$, with right (A, L)-module structure (2.4); given a right (A, L)-module N, the pairing (4.9.4) coincides with the pairing (4.9.3) for the left (A, L)-module $M = \mathrm{Hom}_A(C_L, N)$, with left (A, L)-module structure (2.3). The naturality of the duality isomorphisms via the cap product with the fundamental class, cf. (3.7), establishes at once the following.

Lemma 4.9. *Given a left (A, L)-module M, the following are equivalent.*
(i) *The cap pairing* (4.9.1) *is nondegenerate in the sense that its adjoint*

$$\cap^\sharp\colon \mathrm{Ext}^k_U(M, \mathcal{O}) \longrightarrow \mathrm{Hom}_R(\mathrm{Tor}^U_k(C_L, M), R) \tag{4.9.5}$$

is an isomorphism of R-modules.
(ii) *The cup pairing* (4.9.2) *is nondegenerate in the sense that its adjoint*

$$\cup^\sharp\colon \mathrm{H}^{n-k}(L, \mathrm{Hom}_A(M, \mathcal{O})) \longrightarrow \mathrm{Hom}_R(\mathrm{H}^k(L, M), R) \tag{4.9.6}$$

is an isomorphism of R-modules. □

We note that the adjoint (4.9.5) of (4.9.1) may as well be written

$$\cap^\sharp\colon \mathrm{H}^k(L, \mathrm{Hom}_A(M, \mathcal{O})) \longrightarrow \mathrm{Hom}_R(\mathrm{H}_k(L, C_L \otimes_A M), R). \tag{4.9.7}$$

Likewise the adjoint of the cap pairing (4.9.4) takes the form

$$\cap^\sharp\colon \mathrm{H}^k(L, \mathrm{Hom}_A(N, V)) \longrightarrow \mathrm{Hom}_R(\mathrm{H}_k(L, N), R). \tag{4.9.8}$$

Given a duality (R, A)-Lie algebra with a trace $(\mathcal{O}, t)$ and, furthermore, given a left (A, L)-module M, we shall say that L satisfies *Poincaré duality for M* if

the adjoint (4.9.5) of the cap pairing (4.9.1) is an isomorphism or, equivalently, if the adjoint (4.9.6) of the cup pairing (4.9.2) is an isomorphism of R-modules; given a right (A, L)-module N, we shall say that L satisfies *Poincaré duality for* N if the adjoint (4.9.8) of the cap pairing (4.9.4) is an isomorphism. We note that L satisfies Poincaré duality for a left (A, L)-module M if and only if it satisfies Poincaré duality for the right (A, L)-module $C_L \otimes_A M$; likewise L satisfies Poincaré duality for a right (A, L)-module N if and only if it satisfies Poincaré duality for the left (A, L)-module $\mathrm{Hom}_A(C_L, N)$.

Let W be a connected real smooth manifold. We maintain the notation in (4.7) and do not repeat it. There are various known ways of establishing Poincaré duality in this case: One may introduce a Riemannian metric on W. This induces inner product structures on the constituents of the various de Rham complexes and, via the appropriate Sobolev completions, the canonical pairing $C_L \otimes_{\mathbb{R}} \mathcal{O} \to \mathbb{R}$ passes to a perfect pairing of Hilbert spaces. In the orientable case this is a version of the standard L_2-pairing between functions and densities. From this observation, the nondegeneracy of the cap pairing (4.9.3) may be deduced: The adjoint of the canonical pairing

$$(\omega_A \otimes_U (K \otimes_A M)) \otimes_{\mathbb{R}} \mathrm{Hom}_U(K \otimes_A M, \mathcal{O}) \longrightarrow \omega_A \otimes_U \mathcal{O} \xrightarrow{\iota} \mathbb{R}$$

of chain complexes has the form

$$\omega_A \otimes_U (K \otimes_A M) \longrightarrow \mathrm{Hom}_{\mathbb{R}}(\mathrm{Hom}_U(K \otimes_A M, \mathcal{O}), \mathbb{R});$$

the corresponding space of de Rham currents C is a subspace of the target and the canonical map from $\omega_A \otimes_U (K \otimes_A M)$ to C has dense image and is a chain equivalence. This is similar to the reasoning in de Rham's book [9]. We do not spell out the details since this would not provide any new insight.

Another argument establishing nondegeneracy proceeds by integration against suitable dual cell decompositions of W. Again this would not provide anything new and we refrain from spelling out details.

A third way of establishing nondegeneracy is by means of a Mayer-Vietoris argument. Since this will allow for later generalization we now give a proof for the ordinary de Rham cohomology of manifolds along these lines, within the present framework.

Proposition 4.10. *Under the circumstances of* (4.7)*, suppose that W can be covered by finitely many open contractible sets $V_1, \dots, V_\ell$ such that each non-empty intersection $V_{j_1} \cap \dots \cap V_{j_k}$ is itself contractible. Then, given a left duality (A, L)-module M, the adjoint* (4.9.5) *of the cap pairing* (4.9.1) *(both pairings over $R = \mathbb{R}$) is an isomorphism of real vector spaces.*

For example, every compact manifold may be covered by finitely many open sets in such a way that the hypothesis spelled out above is satisfied. We remind the reader that here a left duality (A, L)-module is just the space of sections of a flat vector bundle on W.

Proof. On an open contractible subset of W, Poincaré duality comes essentially down to the defining property of a trace. The idea of the proof is to reduce the

general case to that of an open contractible manifold by means of a Mayer-Vietoris argument.

The details are as follows. Let $K = K(A, L)$ be the Rinehart complex (2.9.2); it is a projective resolution of A in the category of left $U(A, L)$-modules, cf. what is said in Section 2. Then $K \otimes_A M$, endowed with the left (A, L)-module structure (2.1) (in each degree), is a projective resolution of M in the category of left (A, L)-modules, and the adjoint (4.9.4) is induced by the canonical homomorphism

$$\mathrm{Hom}_U(K \otimes_A M, \mathcal{O}) \longrightarrow \mathrm{Hom}_{\mathbb{R}}(\omega_A \otimes_U (K \otimes_A M), \omega_A \otimes_U \mathcal{O}) \tag{4.10.1}$$

of chain complexes of real vector spaces.

Let $V \subset W$ be an open subspace. The two sides of (4.10.1) are then defined over W and over V; we indicate this by the notation $\mathrm{Hom}_U(K \otimes_A M, \mathcal{O})|_W$, $\mathrm{Hom}_U(K \otimes_A M, \mathcal{O})|_V$, etc. Restriction induces a canonical homomorphism

$$\omega_A \otimes_U (K \otimes_A M)|_W \longrightarrow \omega_A \otimes_U (K \otimes_A M)|_V;$$

furthermore, since $\mathcal{O}$ consists of compactly supported sections, there are canonical injection homomorphisms

$$\mathrm{Hom}_U(K \otimes_A M, \mathcal{O})|_V \longrightarrow \mathrm{Hom}_U(K \otimes_A M, \mathcal{O})|_W$$

and

$$\omega_A \otimes_U \mathcal{O}|_V \longrightarrow \omega_A \otimes_U \mathcal{O}|_W,$$

and these combine to a commutative diagram

$$\begin{array}{ccc} \mathrm{Hom}_U(K \otimes_A M, \mathcal{O})|_V & \longrightarrow & \mathrm{Hom}_{\mathbb{R}}(\omega_A \otimes_U (K \otimes_A M), \omega_A \otimes_U \mathcal{O})|_V \\ \big\downarrow & & \big\downarrow \\ \mathrm{Hom}_U(K \otimes_A M, \mathcal{O})|_W & \longrightarrow & \mathrm{Hom}_{\mathbb{R}}(\omega_A \otimes_U (K \otimes_A M), \omega_A \otimes_U \mathcal{O})|_W \end{array} \tag{4.10.2}$$

of real chain complexes.

For any open subset V of W, we now write

$$C(V) = \mathrm{Hom}_U(K \otimes_A M, \mathcal{O})|_V,\ \ \Omega(V) = \omega_A \otimes_U (K \otimes_A M)|_V,\ \ \Omega^*(V) = \mathrm{Hom}_{\mathbb{R}}(\Omega(V), \mathbb{R}).$$

Given two open sets V_1 and V_2, we obtain two exact sequences of chain complexes

$$0 \to C(V_1 \cap V_2) \to C(V_1) \oplus C(V_2) \to C(V_1 \cup V_2) \to 0 \tag{4.10.3}$$

and

$$\Omega(V_1 \cap V_2) \longleftarrow \Omega(V_1) \oplus \Omega(V_2) \longleftarrow \Omega(V_1 \cup V_2) \longleftarrow 0. \tag{4.10.4}$$

The sequence (4.10.4) is the ordinary one used to derive the Mayer-Vietoris sequence in de Rham theory; even though the morphism from $\Omega(V_1) \oplus \Omega(V_2)$ to $\Omega(V_1 \cap V_2)$ in (4.10.4) is not surjective, (4.10.4) is well known to induce a Mayer-Vietoris sequence

$$\cdots \longleftarrow \mathrm{H}_j\Omega(V_1 \cap V_2) \longleftarrow \mathrm{H}_j\Omega(V_1) \oplus \mathrm{H}_j\Omega(V_2) \longleftarrow \mathrm{H}_j\Omega(V_1 \cup V_2) \longleftarrow \mathrm{H}_{j+1}\Omega(V_1 \cap V_2) \longleftarrow \cdots$$

In fact, the more usual description of this Mayer-Vietoris sequence looks like

$$\to \mathrm{H}^{n-j-1}(V_1\cap V_2)\to \mathrm{H}^{n-j}(V_1\cup V_2)\to \mathrm{H}^{n-j}(V_1)\oplus \mathrm{H}^{n-j}(V_2)\to \mathrm{H}^{n-j}(V_1\cap V_2)\to\cdots$$

where the coefficients are not indicated in notation. The exactness of the Mayer-Vietoris sequence may e. g. be derived from the corresponding sheaf version by an application of the standard device relating Čech cohomology with ordinary (singular) cohomology; cf. e. g. (II.5.6) in [12] (p. 219). It follows that (4.10.4) induces as well a Mayer-Vietoris sequence

$$\cdots\to \mathrm{H}^j\Omega^*(V_1\cap V_2)\to \mathrm{H}^j\Omega^*(V_1)\oplus \mathrm{H}^j\Omega^*(V_2)\to \mathrm{H}^j\Omega^*(V_1\cup V_2)\to \mathrm{H}^{j+1}\Omega^*(V_1\cap V_2)\to$$

On the other hand, it is manifest that (4.10.3) induces a Mayer-Vietoris sequence

$$\cdots\to \mathrm{H}^jC(V_1\cap V_2)\to \mathrm{H}^jC(V_1)\oplus \mathrm{H}^jC(V_2)\to \mathrm{H}^jC(V_1\cup V_2)\to \mathrm{H}^{j+1}C(V_1\cap V_2)\to\cdots$$

This is in fact the Mayer-Vietoris sequence

$$\cdots\to \mathrm{H}^j_{\mathrm{cs}}(V_1\cap V_2)\to \mathrm{H}^j_{\mathrm{cs}}(V_1)\oplus \mathrm{H}^j_{\mathrm{cs}}(V_2)\to \mathrm{H}^j_{\mathrm{cs}}(V_1\cup V_2)\to \mathrm{H}^{j+1}_{\mathrm{cs}}(V_1\cap V_2)\to\cdots$$

in compactly supported de Rham cohomology H_{cs} where again the coefficients are not indicated in notation. The morphism (4.10.1) yields in fact a transformation of functors from $C(\cdot)$ to $\Omega^*(\cdot)$ which, in turn, induces a morphism of Mayer-Vietoris sequences from that involving the functor C to that involving the functor Ω^*. Thus it suffices to prove that

$$C(W)\to\Omega^*(W) \tag{4.10.5}$$

is an isomorphism on cohomology for contractible W. However, for contractible W, the projective A-module M is necessarily free, and it suffices to consider $M=A=C^\infty(W)$. Now $\mathrm{Ext}^k(A,\mathcal{O})\cong \mathrm{H}^k_{\mathrm{cs}}(W)$ is zero for $0\le k<n$ and a copy of the reals for $k=n$. Further, $\mathrm{Tor}^U_k(\omega_A,A)\cong \mathrm{H}^{n-k}(L,A)\cong \mathrm{H}^{n-k}(W)$ which is again zero for $0\le k<n$ and a copy of the reals for $k=n$, and so is $\mathrm{H}^k\Omega^*(W)$. The chain map (4.10.5) identifies the two. This is, in fact, just property (4.4.1) of a trace. □

Remark 4.11. By means of the functor $\mathrm{Tor}^U_k(\omega_A,\cdot)$, the argument for (4.10) reduces the proof of nondegeneracy of the Poincaré duality pairing to that of nondegeneracy of the canonical pairing

$$\mathrm{Tor}^U_k(C_L,M)\otimes_{\mathbb{R}}\mathrm{Ext}^k_U(M,\mathcal{O})\to \mathrm{Tor}^U_0(C_L,\mathcal{O})\cong\mathbb{R}$$

between the indicated Tor- and Ext-groups. Under the circumstances of (4.5), the nondegeneracy of the corresponding pairing (4.5.4) is immediate but, given an arbitrary duality (R,A)-Lie algebra L, in view of the subtle distinction between the ground ring R and the algebra A on which L acts in general non-trivially, nondegeneracy of pairings of the kind (4.9.1) and (4.9.2) is not immediate and perhaps not even always true though we haven't found a counterexample. Moreover,

our proof shows that, under the circumstances of (4.10), given a left duality (A, L)-module M, the group $\mathrm{Tor}_k^U(\omega_A, M)$ may be viewed as one of classes of de Rham *k-cycles* with values in the flat vector bundle corresponding to M. Thus the vector spaces $\mathrm{Tor}_*^U(\omega_A, M)$ play the role of a kind of de Rham *homology vector spaces* for W. For W compact they in fact coincide with the ordinary homology groups with real coefficients. For a general (smooth n-dimensional) manifold W, $\mathrm{Tor}_n^U(\omega_A, A) = \mathrm{H}_n(L, \omega_A)$, and $\mathrm{H}_n(L, \omega_A)$ is one-dimensional, generated by the fundamental class e (introduced after (3.4), whether or not W is compact. This notion of fundamental class comes down to the ordinary one for compact W but, for non-compact W, our theory thus still provides a fundamental class, as does Borel-Moore homology theory.

REMARK 4.12. The argument for (4.10), suitably formalized, can presumably be used to prove Poincaré duality for other Lie-Rinehart algebras (A, L), for example for A a regular affine algebra over a field k and $L = \mathrm{Der}(A)$.

EXAMPLE 4.13. Under the circumstances of (2.12), let $R = \mathbb{R}$, and suppose that $\mathfrak{g}$ is the Lie algebra of a compact (connected) Lie group G of dimension n and that A is the algebra of smooth functions on a smooth manifold W in such a way that the $\mathfrak{g}$-action on A is induced by a G-action on W. Then, cf. (2.12), the cohomology $\mathrm{H}^*(L, M)$ of the $(\mathbb{R}, A)$-Lie algebra $L = A \otimes_{\mathbb{R}} \mathfrak{g}$ with coefficients in any left (A, L)-module M is just the ordinary Lie algebra cohomology $\mathrm{H}^*(\mathfrak{g}, M)$. Furthermore, cf. [11 (3.5)], the Lie algebra cohomology $\mathrm{H}^*(\mathfrak{g}, A)$ is isomorphic to $\mathrm{H}^*(\mathfrak{g}, \mathbb{R}) \otimes_{\mathbb{R}} A^{\mathfrak{g}}$. Thus $\mathcal{O} = A$ and the canonical map

$$t\colon \mathrm{H}^n(\mathfrak{g}, A) \cong \mathrm{H}^n(\mathfrak{g}, \mathbb{R}) \otimes_{\mathbb{R}} A^{\mathfrak{g}} \longrightarrow A^{\mathfrak{g}}$$

to the $\mathfrak{g}$-invariants $A^{\mathfrak{g}}$ is an isomorphism. Presumably $(\mathcal{O}, t)$ yields a trace for L, with $A^{\mathfrak{g}}$ as ground ring. Details have not been verified yet. Moreover, over $A^{\mathfrak{g}}$, the cup pairing

$$\mathrm{H}^j(L, A) \otimes_{A^{\mathfrak{g}}} \mathrm{H}^{n-j}(L, A) \longrightarrow \mathrm{H}^n(L, A) \cong A^{\mathfrak{g}} \tag{4.13.1}$$

manifestly comes down, in the classical Poincaré duality pairing

$$\mathrm{H}^j(\mathfrak{g}, \mathbb{R}) \otimes_{\mathbb{R}} \mathrm{H}^{n-j}(\mathfrak{g}, \mathbb{R}) \longrightarrow \mathrm{H}^n(\mathfrak{g}, \mathbb{R}) \cong \mathbb{R} \tag{4.13.2}$$

in ordinary Lie algebra cohomology, to an extension of scalars from the reals to $A^{\mathfrak{g}}$. Since (4.13.2) is a nondegenerate pairing of finitely generated real vector spaces, (4.13.1) is a nondegenerate pairing of finitely generated free $A^{\mathfrak{g}}$-modules. Thus L satisfies Poincaré duality for $M = A$, with the algebra $A^{\mathfrak{g}}$ of invariants as ground ring. Presumably L satisfies Poincaré duality for any duality (A, L)-module. We note that, unless W reduces to a point, the *nondegeneracy* of (4.13.2) *cannot even be phrased over the reals*. We also note that, when the $\mathfrak{g}$-action is not induced by a G-action, $\mathrm{H}^*(\mathfrak{g}, A)$ need no longer be isomorphic to $\mathrm{H}^*(\mathfrak{g}, \mathbb{R}) \otimes_{\mathbb{R}} A^{\mathfrak{g}}$. A counterexample is given in [11]. Perhaps a counterexample to Poincaré duality may be found along these lines.

Our next aim is to illustrate Poincaré duality with another class of examples. For this purpose we shall need the following.

Lemma 4.14. *Let (A, L) be a duality Lie-Rinehart algebra of rank n over a commutative ring R, let $(\mathcal{O}, t)$ be a trace for L, and let M_1 and M_2 be left duality (A, L)-modules (i.e. left (A, L)-modules which are finitely generated and projective as A-modules). Then L satisfies Poincaré duality for $M = M_1 \oplus M_2$ if and only if it satisfies Poincaré duality for M_1 and for M_2.*

Proof. For $M = M_1 \oplus M_2$, the R-module morphism (4.9.6) may be written in the form

$$\begin{aligned} &\mathrm{H}^{n-k}(L, \mathrm{Hom}_A(M_1, \mathcal{O})) \oplus \mathrm{H}^{n-k}(L, \mathrm{Hom}_A(M_2, \mathcal{O})) \\ &\qquad\longrightarrow \mathrm{Hom}_R(\mathrm{H}^k(L, M_1), R) \oplus \mathrm{Hom}_R(\mathrm{H}^k(L, M_2), R). \end{aligned} \tag{4.14.1}$$

Plainly, (4.14.1) is an isomorphism of R-modules if and only if its constituents

$$\mathrm{H}^{n-k}(L, \mathrm{Hom}_A(M_1, \mathcal{O})) \longrightarrow \mathrm{Hom}_R(\mathrm{H}^k(L, M_1), R)$$

and

$$\mathrm{H}^{n-k}(L, \mathrm{Hom}_A(M_2, \mathcal{O})) \longrightarrow \mathrm{Hom}_R(\mathrm{H}^k(L, M_2), R)$$

are isomorphisms of R-modules. □

Example 4.15. Let again W be a (real) smooth n-dimensional manifold, write $A = C^\infty(W)$, let $\mathcal{F}$ be a foliation on W of codimension $n-k$ with compact leaves, let $L_{\mathcal{F}} \subseteq \mathrm{Vect}(W)$ be the $(\mathbb{R}, A)$-Lie algebra of vector fields tangent to the foliation, and let $\mathcal{O}_{\mathcal{F}}$ be the A-module of sections of the orientation bundle of $\mathcal{F}$. Being a real line bundle, the orientation bundle of $\mathcal{F}$ inherits a canonical flat connection; the latter, in turn, yields a left $(A, \mathrm{Vect}(W))$-module and hence $(A, L_{\mathcal{F}})$-module structure on $\mathcal{O}_{\mathcal{F}}$. Write B for the space of leaves of $\mathcal{F}$ and $C^\infty(B)$ for its algebra of smooth functions, that is, $C^\infty(B)$ is the algebra of smooth functions on W which are constant on the leaves. As an A-module, the dualizing module $C_{L_{\mathcal{F}}} = \mathrm{Hom}_A(\Lambda_A^k L_{\mathcal{F}}, A)$ of $L_{\mathcal{F}}$ (cf. 2.10) is the space of sections of a line bundle on W, and the tensor product $C_{L_{\mathcal{F}}} \otimes_A \mathcal{O}_{\mathcal{F}}$ is the space of densities along the leaves; hence the integration map

$$C_{L_{\mathcal{F}}} \otimes_A \mathcal{O}_{\mathcal{F}} \longrightarrow C^\infty(B), \quad \omega \mapsto \int_{F_b} \omega,$$

induces a map

$$\tau \colon \mathrm{H}^k(L_{\mathcal{F}}, \mathcal{O}_{\mathcal{F}}) \xrightarrow{e \cap \cdot} C_{L_{\mathcal{F}}} \otimes_U \mathcal{O}_{\mathcal{F}} \longrightarrow \mathrm{Map}(B, \mathbb{R}),$$

$e \in \mathrm{H}_k(L_{\mathcal{F}}, C_{\mathcal{F}})$ being the fundamental class of $L_{\mathcal{F}}$, cf. (3.4). Given a density along the leaves ω, the resulting real valued function h on B given by $h(b) = \int_{F_b} \omega$ need not be smooth, though. This happens for example when, roughly speaking, the "volumes" of the leaves "jump", e. g. for a Moebius band with a short leaf. (This observation is due to the referee.)

We now explain an important special case where τ yields a trace (or rather a variant thereof) and where, furthermore, $L_{\mathcal{F}}$ satisfies Poincaré duality with appropriate coefficients. We take as ground ring R the algebra $C^\infty(B)$ rather than

the reals. Suppose that the foliation $\mathcal{F}$ constitutes a fiber bundle $\eta\colon W \to B$ with (compact) fiber F, and write $L_\eta = L_{\mathcal{F}}$ and $\mathcal{O}_\eta = \mathcal{O}_{\mathcal{F}}$. Then τ yields an isomorphism

$$\tau\colon \mathrm{H}^k(L_{\mathcal{F}}, \mathcal{O}_{\mathcal{F}}) \to C^\infty(B) \tag{4.15.1}$$

of $(C^\infty(B))$-modules. For $C_{L_\eta} \otimes_A \mathcal{O}_\eta$ is the space of sections of a trivial line bundle on W; a nowhere vanishing section $\widetilde{\omega}$ is a density along the leaves which, on each fiber (or leaf) F_b, satisfies $\int_{F_b} \widetilde{\omega} \neq 0$, and $f_{\widetilde{\omega}} : b \mapsto \int_{F_b} \widetilde{\omega}$ is a smooth nowhere vanishing function on B (this is not necessarily true when the foliation is not induced from a fiber bundle); when we multiply $\widetilde{\omega}$ by the pull back of $f_{\widetilde{\omega}}$, we obtain a density along the leaves ω which has as image under τ the constant function 1 on B. This shows that τ maps onto $C^\infty(B)$. Injectivity of τ follows from the observation that the customary argument which, given a k-form α on a smooth k-dimensional manifold F whose integral over F vanishes, by integration against suitable paths yields a $(k-1)$-form β such that $d\beta = \alpha$ carries over since integration is compatible with parameters. Thus given a density along the leaves σ such that $\int_{F_b} \widetilde{\sigma} = 0$ for every leaf F_b, there is a $(k-1)$-form β on L_η with values in $\mathcal{O}_\eta$ such that $d\beta = \sigma$. Hence τ is injective. Consequently multiplication of ω by the pull back of a smooth function on B yields the inverse mapping of τ.

Let $\pi\colon P \to B$ be a principal bundle for η, having compact structure group G, let $\zeta\colon V \to F$ be a flat smooth G-vector bundle on F or, equivalently, a G-local system on F, and let $\zeta_W\colon V_W = P \times_G V \to P \times_G F = W$ be its extension to W; this is a fibered vector bundle on W, endowed with a flat connection defined only for smooth vector fields tangent to the fibers, that is, for elements of L_η. Write $M = \Gamma(\zeta_W)$ for its space of sections; it inherits a left (A, L_η)-module structure. In particular, ζ might be the trivial vector bundle on F; in this case, M is a finitely generated free A-module, with the obvious left (A, L_η)-module structure.

Theorem 4.15.3. *The $(\mathbb{R}, A)$-Lie algebra L_η, endowed with the pretrace $(\mathcal{O}_\eta, \tau)$, satisfies Poincaré duality for every such M. More precisely,*

$$\cup\colon \mathrm{H}^\ell(L_\eta, M) \otimes_{C^\infty(B)} \mathrm{H}^{k-\ell}(L_\eta, \mathrm{Hom}_A(M, \mathcal{O}_\eta)) \to \mathrm{H}^k(L_\eta, \mathcal{O}_\eta) \cong C^\infty(B) \tag{4.15.4}$$

is a nondegenerate pairing of finitely generated projective $(C^\infty(B))$-modules. In particular,

$$\cup\colon \mathrm{H}^\ell(L_\eta, A) \otimes_{C^\infty(B)} \mathrm{H}^{k-\ell}(L_\eta, \mathcal{O}_\eta) \to \mathrm{H}^k(L_\eta, \mathcal{O}_\eta) \cong C^\infty(B) \tag{4.15.5}$$

is a nondegenerate pairing of finitely generated projective $(C^\infty(B))$-modules.

We now prepare for the proof of this theorem. To describe the chain complex $\mathrm{Alt}_A(L_\eta, M)$, consider the tangent bundle $\tau_{\mathcal{F}}\colon \mathrm{T}\mathcal{F} \to W$ of $\mathcal{F}$ (which is assumed to arise from a fibre bundle), let $\Lambda\tau_{\mathcal{F}} = \{\Lambda^0\tau_{\mathcal{F}}, \Lambda^1\tau_{\mathcal{F}}, \dots, \Lambda^k\tau_{\mathcal{F}}\}$ be the collection of its graded exterior powers, and consider the system $\mathrm{HOM}(\Lambda\tau_{\mathcal{F}}, \zeta_W)$ of smooth vector bundles on W where HOM stands for morphisms of vector bundles. Write $\tau_F\colon \mathrm{T}F \to F$ for the tangent bundle of F and let $A_F = C^\infty(F)$, $L_F = \mathrm{Vect}(F)$, $M_F = \Gamma(\zeta)$; each $\mathrm{HOM}(\Lambda^j\tau_{\mathcal{F}}, \zeta_W)$ may clearly be written

$$P \times_G \mathrm{HOM}(\Lambda^j\tau_F, \zeta)\colon P \times_G \mathrm{HOM}(\Lambda^j\mathrm{T}F, V) \to P \times_G F = W,$$

and the chain complex $\mathrm{Alt}_A(L_\eta, M)$ is obtained from $(C^\infty(P)) \otimes_{\mathbb{R}} \mathrm{Alt}_{A_F}(L_F, M_F)$ when G-invariants are taken. On the other hand, $\mathrm{Alt}_{A_F}(L_F, M_F)$ is just the de Rham complex of the fibre F with coefficients in the flat vector bundle ζ, and its cohomology $\mathrm{H}^*(F, \zeta)$ inherits a G-module structure; consider the associated smooth vector bundle

$$h^*(\pi, \zeta) \colon P \times_G \mathrm{H}^*(F, \zeta) \to B.$$

Its space of sections $\Gamma(h^*(\pi, \zeta))$ boils down to the space of G-invariants of

$$(C^\infty(P)) \otimes_{\mathbb{R}} \mathrm{H}^*(F, \zeta) \cong \mathrm{H}^*((C^\infty(P)) \otimes_{\mathbb{R}} \mathrm{Alt}_{A_F}(L_F, M_F)).$$

Further, it is manifest that, as $(C^\infty(B))$-modules, the cohomology groups $\mathrm{H}^*(L_\eta, M)$ and $\mathrm{H}^*(L_\eta, \mathrm{Hom}_A(M, \mathcal{O}_\eta))$ are projective. The proof of (4.15.3) now relies on the following.

Lemma 4.15.6. *The canonical map from*

$$\mathrm{H}^*(L_\eta, M) \cong \mathrm{H}^*\left(((C^\infty(P)) \otimes_{\mathbb{R}} \mathrm{Alt}_{A_F}(L_F, M_F))^G\right)$$

to the space of sections of $h^*(\pi, \zeta)$ is an isomorphism of $(C^\infty(B))$-modules.

Proof. Write $C = (C^\infty(P)) \otimes_{\mathbb{R}} \mathrm{Alt}_{A_F}(L_F, M_F)$ for short; the group G being compact, the standard argument involving invariant integration on G shows that the canonical map from $\mathrm{H}^*(C^G)$ to $(\mathrm{H}^* C)^G$ is an isomorphism, that is to say, the canonical map

$$\mathrm{H}^*(L_\eta, M) \to ((C^\infty(P)) \otimes_{\mathbb{R}} \mathrm{H}^*(F, \zeta))^G = \Gamma(h^*(\pi, \zeta))$$

is an isomorphism of $(C^\infty(B))$-modules. □

Proof of Theorem (4.15.3). Lemma 4.15.6 reduces the question whether the pairing (4.15.4) is nondegenerate to ordinary Poincaré duality for F with coefficients in ζ. Alternatively, we may cover F by finitely many G-invariant open contractible subsets $V_1, \ldots, V_\ell$ such that each non-empty intersection $V_{j_1} \cap \cdots \cap V_{j_k}$ is itself contractible and, on any such non-empty intersection, work with those sections of the orientation bundle of the corresponding foliation which, on each leaf, are compactly supported. Poincaré duality may then be established by an argument generalizing that for (4.10) above. □

Let (A, L) be a Lie-Rinehart algebra over a general commutative ring R. A left (A, L)-module M will be said to be an *induced* left (A, L)-module provided there is an A^L-module M' such that, as a left (A, L)-module, M coincides with $A \otimes_{A^L} M'$ where $A \otimes_{A^L} M'$ is endowed with the obvious left (A, L)-module structure induced by that on A.

Corollary 4.15.7. *The $(\mathbb{R}, A)$-Lie algebra $L = L_\eta$, endowed with the pretrace $(\mathcal{O}_\eta, \tau)$, satisfies Poincaré duality for every induced left (A, L_η)-module M of the kind $M = A \otimes_{A^L} M'$ where M' is a finitely generated projective A^L-module. That is to say,*

$$\cup \colon \mathrm{H}^\ell(L_\eta, M) \otimes_{C^\infty(B)} \mathrm{H}^{k-\ell}(L_\eta, \mathrm{Hom}_A(M, \mathcal{O}_\eta)) \to \mathrm{H}^k(L_\eta, \mathcal{O}_\eta) \cong C^\infty(B) \qquad (4.15.8)$$

is a nondegenerate pairing of finitely generated projective $(C^\infty(B))$-modules.

Proof. By (4.15.3), the pairing (4.15.4) is nondegenerate when M is a finitely generated free A-module, endowed with the obvious left (A,L)-module structure. The general case is then a consequence of Lemma 4.14. $\square$

5. Extensions of Lie-Rinehart algebras and Poincaré duality

In this section we shall show that Poincaré duality is preserved under extensions of Lie-Rinehart algebras. In Section 7 below, this will enable us to show that, in certain cases, Poisson (co)homology satisfies Poincaré duality.

Let

$$0 \longrightarrow L' \longrightarrow L \longrightarrow L'' \longrightarrow 0 \tag{5.1}$$

be an extension of duality (R,A)-Lie algebras; let $k = \operatorname{rank}(L')$, $m = \operatorname{rank}(L'')$, $n = k+m = \operatorname{rank}(L)$. Then L' is an ordinary Lie algebra in the category of A-modules which, as an A-module, is finitely generated and projective of constant rank k. Hence, cf. (4.5), it has a canonical trace $(\mathcal{O}', t')$ and satisfies *classical* Poincaré duality for every left L'-module, cf. (4.5) above.

Proposition 5.2. *A trace for L'' induces a trace for L.*

The precise relationships between the traces for L and L'' will be given in (5.2.1) and (5.2.2) below.

Proof. Let $(\mathcal{O}_{L''}, t'')$ be a trace for L'', with $t'' \colon C_{L''} \otimes_{U''} \mathcal{O}_{L''} \xrightarrow{\cong} R$, and, cf. (4.6), let $V'' = C_{L''} \otimes_A \mathcal{O}_{L''}$, endowed with the right (A,L'')-module structure (2.3). Furthermore, let

$$\mathcal{O}_L = \operatorname{Hom}_A(C_L, C_{L''}) \otimes_A \mathcal{O}_{L''} \tag{5.2.1}$$

with left (A,L)-module structure on $\operatorname{Hom}_A(C_L, C_{L''})$ given by (2.3) and with the ordinary tensor product left (A,L)-module structure on $\mathcal{O}_L$, and write V for V'', viewed as a right (A,L)-module via the projection from L to L''. Then the canonical maps

$$C_L \otimes_U (\operatorname{Hom}_A(C_L, C_{L''}) \otimes_A \mathcal{O}_{L''}) \longrightarrow (C_L \otimes_A \operatorname{Hom}_A(C_L, C_{L''})) \otimes_U \mathcal{O}_{L''}$$

and

$$(C_L \otimes_A \operatorname{Hom}_A(C_L, C_{L''})) \otimes_U \mathcal{O}_{L''} \longrightarrow C_{L''} \otimes_{U''} \mathcal{O}_{L''}$$

are isomorphisms (of R-modules) and the composite

$$t \colon C_L \otimes_U \mathcal{O}_L \longrightarrow C_{L''} \otimes_{U''} \mathcal{O}_{L''} \xrightarrow{t''} R \tag{5.2.2}$$

is an isomorphism of R-modules which in fact yields a trace for L. To verify the defining property of a trace, we use (4.6)(ii): Let N be a right duality (A,L)-module. Then

$$\operatorname{Hom}_U(N, V) \cong \operatorname{Hom}_{U''}(N \otimes_{U'} A, V'')$$

and

$$\operatorname{Hom}_R(N \otimes_U A, V \otimes_U A) \cong \operatorname{Hom}_R((N \otimes_{U'} A) \otimes_{U''} A, V'' \otimes_{U''} A).$$

Since $(\mathcal{O}_{L''}, t'')$ is a trace for L'' and since N is a right duality (A, L)-module, by (4.6) (i), the canonical morphism

$$\mathrm{Hom}_{U''}(N \otimes_{U'} A, V'') \longrightarrow \mathrm{Hom}_R((N \otimes_{U'} A) \otimes_{U''} A, V'' \otimes_{U''} A)$$

is an isomorphism. Consequently the canonical morphism

$$\mathrm{Hom}_U(N, V) \longrightarrow \mathrm{Hom}_R(N \otimes_U A, V \otimes_U A)$$

is an isomorphism. Since N was arbitrary, by (4.6) (ii), $(\mathcal{O}, t)$ indeed yields a trace for (A, L). □

It is readily seen that, for a left (A, L)-module M, the canonical L-action on the chain complex $\mathrm{Alt}_A(L', M)$ (over the ground ring R) makes $\mathrm{Alt}_A(L', M)$ into a chain complex in the category of left (A, L)-modules, and the cohomology $\mathrm{H}^*(L', M)$ thus inherits a left (A, L'')-module structure. Likewise, for a right (A, L)-module N, the (A, L)-action on the chain complex $N \otimes_A K(A, L')$ (over the ground ring R) given by (2.4) in each degree makes $N \otimes_A K(A, L')$ into a chain complex in the category of right (A, L)-modules, and the homology $\mathrm{H}_*(L', N)$ inherits a right (A, L'')-module structure.

Theorem 5.3. *Let $(\mathcal{O}'', t'')$ be a trace for L'', and let $(\mathcal{O}, t)$ be the corresponding trace for L given by* (5.2)*. Given a right (A, L)-module N, if the (R, A)-Lie algebra L'' satisfies Poincaré duality for the right (A, L'')-modules*

$$\mathrm{H}_0(L', N), \mathrm{H}_1(L', N), \ldots, \mathrm{H}_k(L', N),$$

then the (R, A)-Lie algebra L satisfies Poincaré duality for the right (A, L)-module N.

We note that the theorem may also be phrased for a left (A, L)-module (instead of the right (A, L)-module N) but the wording would be technically more involved. The translation is by means of the standard device: Given a left (A, L)-module M, let $N = C_L \otimes_A M$, with right (A, L)-module structure (2.4); given a right (A, L)-module N, let $M = \mathrm{Hom}_A(C_L, N)$, with left (A, L)-module structure (2.3). We leave the details of the translation of the wording of (5.3) into a statement involving left modules rather than right ones to the reader.

Proof. Let $V = C_L \otimes_A O$, with right (A, L)-module structure (2.4). Consider the cohomology change of rings spectral sequence $(\mathrm{E}_r^{p,q}(\mathrm{Hom}_A(N, V)), d_r)$ for the canonical surjection $U(A, L) \to U(A, L'')$, with coefficients in the left (A, L)-module $\mathrm{Hom}_A(N, V)$, with left (A, L)-module structure (2.3). See $(2)_4$ in (XVI.5) (p. 349) of [7]. This spectral sequence has

$$\mathrm{E}_2^{p,q}(\mathrm{Hom}_A(N, V)) = \mathrm{H}^p(L'', \mathrm{H}^q(L', \mathrm{Hom}_A(N, V))), \quad p, q \geq 0.$$

Likewise, with coefficients in the right (A, L)-module N, the homology change of rings spectral sequence $(\mathrm{E}^r_{p,q}(N), d^r)$ for the canonical surjection $U(A, L) \to U(A, L'')$ ($(2)_2$ in (XVI.5) (p. 348) of [7]) has

$$\mathrm{E}^2_{p,q}(N) = \mathrm{H}_p(L'', \mathrm{H}_q(L', N)), \quad p, q \geq 0.$$

Moreover, for every $q \geq 0$, the cap pairing

$$\mathrm{H}_q(L', N) \otimes_A \mathrm{H}^q(L', \mathrm{Hom}_A(N, V)) \longrightarrow V \tag{5.3.1}$$

for the ordinary A-Lie algebra L' is a pairing of A-modules and, by classical duality for L', cf. (4.5), the adjoint

$$\mathrm{H}^q(L', \mathrm{Hom}_A(N, V)) \longrightarrow \mathrm{Hom}_A(\mathrm{H}_q(L', N), V) \tag{5.3.2}$$

of (5.3.1) is an isomorphism of A-modules. Furthermore, for $q \geq 0$, $\mathrm{H}_q(L', N)$ and $\mathrm{H}^q(L', \mathrm{Hom}_A(N, V))$ inherit right- and left (A, L'')-module structures, respectively, in such a way that, when $\mathrm{Hom}_A(\mathrm{H}_q(L', N), V)$ is endowed with the left (A, L'')-module structure (2.3), (5.3.2) is in fact an isomorphism of left (A, L'')-modules. Consequently, given $p, q \geq 0$, the cap pairing

$$\mathrm{H}_p(L'', \mathrm{H}_q(L', N)) \otimes_R \mathrm{H}^p(L'', \mathrm{H}^q(L', \mathrm{Hom}_A(N, V))) \longrightarrow \mathrm{H}_0(L'', V'') \cong R \tag{5.3.3}$$

for L'', with reference to the bilinear pairing (5.3.1), amounts to the corresponding Poincaré duality pairing (4.9.4) (with L'' and $\mathrm{H}_q(L', N)$ instead of L and N, respectively); it has the form

$$\mathrm{H}_p(L'', \mathrm{H}_q(L', N)) \otimes_R \mathrm{H}^p(L'', \mathrm{Hom}_A(\mathrm{H}_q(L', N), V)) \longrightarrow \mathrm{H}_0(L'', V'') \cong R. \tag{5.3.4}$$

By hypothesis, L'' satisfies Poincaré duality for the right (A, L'')-modules

$$\mathrm{H}_0(L', N), \ldots, \mathrm{H}_k(L', N).$$

Hence, given $p, q \geq 0$, the adjoint

$$\mathrm{H}^p(L'', \mathrm{Hom}_A(\mathrm{H}_q(L', N), V)) \longrightarrow \mathrm{Hom}_R(\mathrm{H}_p(L'', \mathrm{H}_q(L', N)), R)$$

of (5.3.4) is an isomorphism of R-modules whence the adjoint

$$\mathrm{H}^p(L'', \mathrm{H}^q(L', \mathrm{Hom}_A(N, V))) \longrightarrow \mathrm{Hom}_R(\mathrm{H}_p(L'', \mathrm{H}_q(L', N)), R) \tag{5.3.5}$$

of the cap pairing (5.3.3) is an isomorphism of R-modules.

The description of the change of rings spectral sequences in [7] involves suitable bicomplexes arising from appropriate resolutions. For the present circumstances, we now indicate briefly a different description which is more appropriate for our purposes: The projection from L to L'' induces a surjection from $\Lambda_A L$ onto $\Lambda_A L''$, whence the degree filtration of $\Lambda_A L''$ induces a filtration $F_0 \subseteq F_1 \subseteq \ldots$ of $\Lambda_A L$ where an element $\alpha_1 \wedge \alpha_2 \wedge \cdots \wedge \alpha_k$ of $\Lambda_A L$ lies in F_p if among the $\alpha_1, \alpha_2, \ldots, \alpha_k$ in L at most p have non-zero image in L''. This filtration induces a filtration of the Rinehart complex $K(A, L)$ and, therefore, filtrations of the chain complexes

$$N \otimes_U K(A, L) \quad \text{and} \quad \mathrm{Hom}_U(K(A, L), \mathrm{Hom}_A(N, V)).$$

The spectral sequences $(\mathrm{E}^r_{p,q}(N), d^r)$ and $(\mathrm{E}_r^{p,q}(\mathrm{Hom}_A(N, V)), d_r)$ arise from these filtrations. The cap pairing between the relevant chain complexes is compatible

with the filtrations and hence, in view of standard spectral sequence multiplicative properties, induces a pairing

$$(\mathrm{E}^r_{p,q}(N), d^r) \otimes_R (\mathrm{E}_r^{p,q}(\mathrm{Hom}_A(N, V)), d_r) \longrightarrow \mathrm{E}^r_{0,0}(V), \quad p, q \geq 0, \ \ r \geq 2, \tag{5.3.6}$$

of spectral sequences. However, since t is a trace for L, the canonical morphisms

$$\mathrm{H}_0(L'', \mathrm{H}_0(L', V)) = \mathrm{E}^2_{0,0}(V) \longrightarrow \ldots \longrightarrow \mathrm{E}^r_{0,0}(V)$$

are all isomorphisms and the trace induces canonical isomorphisms between the $\mathrm{E}^r_{0,0}(V)$'s and the ground ring R. Consequently the pairing (5.3.6) of spectral sequences may be written

$$(\mathrm{E}^r_{p,q}(N), d^r) \otimes_R (\mathrm{E}_r^{p,q}(\mathrm{Hom}_A(N, V)), d_r) \longrightarrow R, \quad p, q \geq 0, \ \ r \geq 2. \tag{5.3.7}$$

Furthermore, since (5.3.5) is an isomorphism, the pairing (5.3.5) has the property that, for $r = 2$, the adjoint

$$\mathrm{E}_2^{p,q}(\mathrm{Hom}_A(N, V)) \longrightarrow \mathrm{Hom}_R(\mathrm{E}^2_{p,q}(N), R)$$

is an isomorphism of bigraded R-modules. Hence the adjoint

$$(\mathrm{E}_r^{p,q}(\mathrm{Hom}_A(N, V)), d_r) \longrightarrow \mathrm{Hom}_R((\mathrm{E}^r_{p,q}(N), d^r), R) = \big(\mathrm{Hom}_R((\mathrm{E}^r_{p,q}(N), R), (d^r)^*\big)$$

of the pairing (5.3.7) is an isomorphism of spectral sequences in the category of R-modules. Thus the adjoint

$$\mathrm{E}_\infty^{p,q}(\mathrm{Hom}_A(N, V)) \longrightarrow \mathrm{Hom}_R(\mathrm{E}^\infty_{p,q}(N), R), \quad p, q \geq 0,$$

of the induced pairing

$$\mathrm{E}^\infty_{p,q}(N) \otimes_R \mathrm{E}_\infty^{p,q}(\mathrm{Hom}_A(N, V)) \longrightarrow R \tag{5.3.8}$$

is an isomorphism of bigraded R-modules. Consequently, for every $\ell \geq 0$, the adjoint

$$\mathrm{H}^\ell(L, \mathrm{Hom}_A(N, V)) \longrightarrow \mathrm{Hom}_R(\mathrm{H}_\ell(L, N), R) \tag{5.3.9}$$

of the cap pairing

$$\mathrm{H}_\ell(L, N) \otimes_R \mathrm{H}^\ell(L, \mathrm{Hom}_A(N, V)) \longrightarrow \mathrm{H}_0(L, V) \cong R \tag{5.3.10}$$

is an isomorphism of R-modules. This proves the claim. □

We now illustrate the last result with a special class of examples. Let the ground ring be that of the reals, $\mathbb{R}$, and let A be the algebra of smooth functions on a smooth manifold W. Recall that a Lie algebroid on W, cf. e. g. [19], determines a duality $(\mathbb{R}, A)$-Lie algebra, and every duality $(\mathbb{R}, A)$-Lie algebra arises in this way. By (5.2), given an $(\mathbb{R}, A)$-Lie algebra L which arises from a transitive Lie algebroid, when $(\mathcal{O}_A, t_A)$ refers to the canonical trace for $\mathrm{Der}(A) = \mathrm{Vect}(W)$, cf. (4.7), the canonical isomorphism between $C_L \otimes_A (Q_L \otimes_A \mathcal{O}_A)$ and $\omega_A \otimes_A \mathcal{O}_A$ yields a trace

$$(\mathcal{O}_L, t_L) = (Q_L \otimes_A \mathcal{O}_A, t_L)$$

for L.

Theorem 5.4. *Every (duality) Lie-Rinehart algebra (A, L) over the reals arising from a transitive Lie algebroid on a smooth manifold satisfies Poincaré duality for any duality (A, L)-module.*

Proof. This is an immediate consequence of (5.3), combined with Poincaré duality for the $(\mathbb{R}, A)$-Lie algebra of smooth vector fields on the manifold which comes into play, i. e. with standard Poincaré duality in ordinary de Rham cohomology of smooth manifolds, cf. (4.10). □

For an *integrable* transitive Lie algebroid, i. e. one arising from a principal bundle, this result follows of course from ordinary Poincaré duality for the total space of the principal bundle. The statement of the theorem is more general, though, since there are transitive Lie algebroids which do not arise from a principal bundle [1].

6. The Picard group and modular class of a Lie-Rinehart algebra

Let L be an (R, A)-Lie algebra. We shall denote the A-module of derivations of L in A, that is, that of A-valued 1-cocycles, by $\mathrm{Der}(L, A)$. We remind the reader that $\phi \in \mathrm{Hom}_A(L, A)$ is called a *derivation* provided

$$\phi([\alpha, \beta]) = \alpha\phi(\beta) - \beta\phi(\alpha), \quad \alpha, \beta \in L.$$

Let M be an A-module. Recall that an *L-connection* on M may be described as a pairing $L \otimes_R M \to M$ of R-modules, written $(\alpha, x) \mapsto \alpha(x)$, such that

$$\begin{aligned} \alpha(ax) &= \alpha(a)x + a\alpha(x), \\ (a\alpha)(x) &= a(\alpha(x)), \end{aligned}$$

where $a \in A$, $x \in M$, $\alpha \in L$; an L-connection on M which is actually a *left* (A, L)-module structure is also said to be *flat*. See [15] (2.16) for historical remarks on these algebraic notions of connection etc. Recall [15] (Section 2) that any projective A-module has an L-connection. Let now M be a projective rank one A-module admitting a flat L-connection, that is, a left (A, L)-module structure. We fix such a structure $L \to \mathrm{End}_R(M)$ and denote by α_M the operator on M which is the image of $\alpha \in L$. The next statement generalizes the well known fact that, for a flat line bundle on a smooth manifold W, the group of de Rham 1-cocycles $Z^1(W, \mathbb{R})$ acts faithfully and transitively on the space of flat connections.

Proposition 6.1. *The A-module M being projective of rank one, the assignment to $\phi \in \mathrm{Der}(L, A)$ and $\alpha \in L$ of*

$$\alpha_{M,\phi} \colon M \to M, \quad \alpha_{M,\phi}(m) = \alpha_M(m) + \phi(\alpha)m, \quad m \in M,$$

yields a faithful and transitive action of $\mathrm{Der}(L, A)$ *on the set of left (A, L)-module structures on M.*

Proof. This is straightforward and left to the reader. □

For $M = A$, endowed with the L-action which is part of the (R, A)-Lie algebra structure (of L), the statement of (6.1) comes down to the well known fact that every derivation ϕ of L in A induces a left (A, L)-module structure on A which is entirely characterized by $\alpha(1) = \phi(\alpha)$, where $\alpha \in L$. Given a derivation ϕ of

L in A, the general statement of (6.1) is then obtained when M is canonically identified with $M \otimes_A A$, the tensor product $M \otimes_A A$ being endowed with the left (A, L)-module structure (2.1).

What corresponds to the group of gauge transformations for a line bundle is now the group of units $A^\times$ in A; indeed, when A is the algebra of smooth functions on a smooth manifold, $A^\times$ is precisely the multiplicative group of nowhere-zero functions, that is, that of functions with values in the multiplicative group $\mathbb{R}^\times$ of non-zero real numbers. In our general situation, the algebra $\mathrm{End}_A(M)$ is canonically isomorphic to A, whence the group $\mathrm{Aut}_A(M)$ amounts to $A^\times$. It acts on $\mathrm{Der}(L, A)$ by the association

$$\phi \mapsto {}^u\phi, \quad {}^u\phi = \phi - u^{-1}du, \quad \phi \in \mathrm{Der}(L, A),\ u \in A^\times,$$

where as usual $du \in \mathrm{Der}(L, A)$ refers to the 1-coboundary $du(\alpha) = \alpha(u)$, for $\alpha \in L$. We denote the space of $A^\times$-orbits in $\mathrm{Der}(L, A)$ by $\mathcal{H}^1(L, A)$.

Proposition 6.2. *The space of $A^\times$-orbits $\mathcal{H}^1(L, A)$ inherits an abelian group structure from that of* $\mathrm{Der}(L, A)$*.*

Remark 6.3. In the gauge theory situation, $\mathcal{H}^1(L, A)$ is just the abelian group of flat line bundles which are trivial as line bundles. Given a smooth real manifold W, this group amounts to the first cohomology group $\mathrm{H}^1(W, \mathbb{R}^\times_+)$ with coefficients in the multiplicative group of positive real numbers $\mathbb{R}^\times_+$. In fact, the holonomy induces a map from the space of flat connections on a trivial line bundle on W to the space

$$\mathrm{Hom}(\pi_1(W), \mathbb{R}^\times_+) \cong \mathrm{H}^1(W, \mathbb{R}^\times_+),$$

and this map induces an isomorphism from $\mathcal{H}^1(L, A)$ onto $\mathrm{H}^1(W, \mathbb{R}^\times_+)$, A and L being the algebra of smooth functions and Lie algebra of smooth vector fields, respectively, on W. Since the structure group, $\mathbb{R}^\times_+$, is abelian, this map is just given by the assignment to a smooth 1-form α of the homomorphism $\phi_\alpha \colon \pi_1(W) \to \mathbb{R}^\times_+$ given by

$$\phi_\alpha[c] = \exp \int_c \alpha,$$

perhaps with a minus sign (depending on the choice of convention), where c is a closed curve in W representing an element of $\pi_1(W)$. Thus, for an general Lie-Rinehart algebra (A, L), the group $\mathcal{H}^1(L, A)$ generalizes the cohomology group $\mathrm{H}^1(W, \mathbb{R}^\times_+)$. Via the logarithm, this group amounts to the more traditional $\mathrm{H}^1(W, \mathbb{R})$.

Proof of (6.2). Let $\phi, \phi', \psi, \psi' \in \mathrm{Der}(L, A)$, and suppose that there are $u, v \in A^\times$ such that

$$\phi' = \phi - u^{-1}du, \quad \psi' = \psi - v^{-1}dv.$$

Then

$$\phi' + \psi' = \phi + \psi - (uv)^{-1}d(uv).$$

Hence the space of $A^\times$-orbits constitutes an abelian group. □

Next we realize this group, in fact, construct a larger group containing it as a subgroup. Recall that $\mathrm{Pic}(A)$ is the abelian group of isomorphism classes of

projective rank one A-modules, the group structure being induced by the tensor product. We denote by $\mathrm{Pic}^{\mathrm{flat}}(L,A)$ the set of isomorphism classes of left (A,L)-modules which are projective of rank one as A-modules. We mention in passing that we have written $\mathrm{Pic}^{\mathrm{flat}}(L,A)$ rather than just $\mathrm{Pic}(L,A)$ since, when L has suitable additional structure, there is presumably also a group of projective of rank one A-modules with "harmonic" L-connections generalizing the group of line bundles with harmonic connection; this group is then to be denoted by $\mathrm{Pic}(L,A)$. The operation of tensor product endows $\mathrm{Pic}^{\mathrm{flat}}(L,A)$ with an abelian group structure. The inverse of the class of such an (A,L)-module M is given by the class of the (A,L)-module $\mathrm{Hom}_A(M,A)$, with left (A,L)-module structure given by (2.2). The zero element of this group is the class of the free rank one A-module, with L-action given by $\alpha(b)=0$, where $\alpha\in L$ and b is a basis element. Notice in particular that an automorphism of a left (A,L)-module which is projective of rank one as an A-module may be viewed as a gauge transformation.

Proposition 6.4. *For a free rank one A-module M, with basis b, the assignment to $\phi\in\mathrm{Der}(L,A)$ and $\alpha\in L$ of*

$$\alpha_{M,\phi}\colon M\to M,\quad \alpha_{M,\phi}(b)=\phi(\alpha)b$$

identifies the kernel of the obvious forgetful map from $\mathrm{Pic}^{\mathrm{flat}}(L,A)$ to $\mathrm{Pic}(A)$ with the group $\mathcal{H}^1(L,A)$.

Proof. Given $\phi\in\mathrm{Der}(L,A)$, write M_ϕ for the free rank one A-module with basis b and left (A,L)-structure given by

$$\alpha_{M,\phi}\colon M\to M,\quad \alpha\in L.$$

For $\phi,\psi\in\mathrm{Der}(L,A)$, on $M_\phi\otimes_A M_\psi$, the corresponding left (A,L)-module structure (2.1) is given by

$$\alpha(b\otimes_A b)=(\alpha_{M,\phi}b)\otimes_A b+b\otimes_A(\alpha_{M,\psi}b)=(\phi(\alpha)b)\otimes_A b+b\otimes_A(\psi(\alpha)b)$$

which amounts to

$$\alpha_{M\otimes_A M,\phi+\psi}\colon(b\otimes_A b)\longmapsto(\phi(\alpha)+\psi(\alpha))(b\otimes_A b)$$

whence the assertion. $\square$

Summing up and combining with [15] (2.15.1), we obtain the following.

Theorem 6.5. *The assignment to a projective rank one A-module of its characteristic class in $\mathrm{H}^2(L,A)$ yields a map from $\mathrm{Pic}(A)$ to $\mathrm{H}^2(L,A)$ which fits into the exact sequence*

$$0\to\mathcal{H}^1(L,A)\to\mathrm{Pic}^{\mathrm{flat}}(L,A)\to\mathrm{Pic}(A)\to\mathrm{H}^2(L,A). \tag{6.5.1}$$

We now suppose that, as an A-module, the (R,A)-Lie algebra $\mathrm{Der}(A)$ is finitely generated and projective of constant rank. For simplicity, we then refer to A as being *regular*. Extending notation already used in (4.7), we denote the dualizing

module $\mathrm{Hom}_A(\Lambda_A^{\mathrm{top}}\mathrm{Der}(A), A)$ of $\mathrm{Der}(A)$ by ω_A. Here Λ_A^{top} refers to the highest non-zero exterior power of $\mathrm{Der}(A)$; its degree equals the rank of $\mathrm{Der}(A)$. Our notion of regularity is consistent with standard terminology in algebraic geometry: When the module D_A of formal differentials is finitely generated and projective of constant rank as an A-module, $\mathrm{Der}(A)$ will likewise have this property, and the requirement that D_A be finitely generated and projective of constant rank essentially amounts to the usual notion of regularity in algebraic geometry. Further, we can then identify ω_A with the highest non-zero exterior power $\Lambda_A^{\mathrm{top}}\mathrm{D}_A$ of D_A; in algebraic geometry, a closely related object is called the "canonical sheaf" and denoted by ω with an appropriate subscript whence the notation. Likewise, cf. (4.7), when A is the algebra of smooth functions on a smooth real manifold W, ω_A amounts to the highest non-zero exterior power of the space of sections of the cotangent bundle of W. As observed in (2.8), the operation of Lie derivative (2.6.2) endows the projective rank one A-module ω_A with a right $(A, \mathrm{Der}(A))$-module structure.

Let L be a duality (R, A)-Lie algebra of rank n, and let C_L $(= \Lambda_A^n L^*)$ be its dualizing module. Let

$$Q_L = \mathrm{Hom}_A(C_L, \omega_A); \tag{6.6}$$

this is a projective rank one A-module which has a canonical left (A, L)-module structure given by (2.3). We refer to the class $[Q_L]$ of Q_L in $\mathrm{Pic}^{\mathrm{flat}}(L, A)$ as the *modular class* of L. A version of the module Q_L occurred already in Section 5: when the structure map $L \to \mathrm{Der}(A)$ is surjective, and when $\mathrm{Der}(A)$ has a trace, written $(\mathcal{O}_A, t_A)$, the module written $\mathrm{Hom}_A(C_L, C_{L''})$ in (5.2.1) is precisely of the kind Q_L, where now $L'' = \mathrm{Der}(A)$; letting $\mathcal{O}_L = Q_L \otimes_A \mathcal{O}_A$, with left (A, L)-module structure (2.1), we then obtain a trace for L by means of the canonical isomorphism

$$C_L \otimes_U \mathcal{O}_L \longrightarrow \omega_A \otimes_{U''} \mathcal{O}_A \xrightarrow{t_A} R,$$

cf. (5.2.2).

When A and L are the algebra of smooth real functions and $(\mathbb{R}, A)$-Lie algebra of smooth vector fields, respectively, on a smooth real manifold W, as an A-module,

$$Q_L = \mathrm{Hom}_A(C_L, \omega_A) = \mathrm{Hom}_A(\omega_A, \omega_A),$$

that is, Q_L is free of rank one, generated by the identity map. Furthermore, its left (A, L)-module structure is "trivial" in the sense that it has a basis, here the identity map, which remains invariant under the L-action; in fact, this means that Q_L and A are isomorphic even as left (A, L)-modules, and the modular class $[Q_L]$ is trivial.

REMARK 6.7. Let $R = \mathbb{R}$, let W be a smooth real manifold, let A be its algebra of smooth real functions, and let L be the $(\mathbb{R}, A)$-Lie algebra coming from a Lie algebroid on W. The group $\mathcal{H}^1(L, A)$ is the space of $A^\times$-orbits in $\mathrm{Der}(L, A)$, and the passage to $A^\times$-orbits amounts to the identification of two 1-cocycles ϕ and ψ whenever $\phi - \psi = u^{-1}du$ for some $u \in A^\times$. Under the present circumstances, a unit u boils down to a nowhere vanishing function f, and

$$u^{-1}du = f^{-1}df = d\log|f|$$

is a coboundary. Hence the identity map of $\mathrm{Der}(L,A)$ induces a canonical map from $\mathcal{H}^1(L,A)$ to $\mathrm{H}^1(L,A)$. Moreover, the group $\mathrm{Pic}(A)$ is just the group $\mathrm{Hom}(\pi_1(W),\mathbb{Z}/2)\cong \mathrm{H}^1(W,\mathbb{Z}/2)$. Consequently, the square $[Q_L]^2$ of the modular class in $\mathrm{Pic}^{\mathrm{flat}}(L,A)$ lies in $\mathcal{H}^1(L,A)$ and hence maps to $\mathrm{H}^1(L,A)$. In [10], this image, divided by 2, is taken as the definition of the modular class (of the corresponding Lie algebroid). One of the disadvantages of such a definition is this: If M is a projective rank one A-module with a left (A,L)-structure, as an A-module, its tensor square $M\otimes_A M$ is free, and hence the tensor square left (A,L)-structure is given by a derivation ϕ of L in A as in (6.4); when we then endow the free rank one A-module with the left (A,L)-structure which corresponds to the derivation $\frac{1}{2}\phi$ of L in A, M and the free rank one A-module with the left (A,L)-structure given by $\frac{1}{2}\phi$ would determine the same element in $\mathrm{H}^1(L,A)$; the two left (A,L)-modules are, of course, distinguished in $\mathrm{Pic}^{\mathrm{flat}}(L,A)$. Another disadvantage is that such a definition will not work over other ground rings like e. g. that of the complex numbers, unless the projective rank one A-module under consideration has finite order in the Picard group.

Remark 6.8. Under the circumstances of (6.7), when L is the $(\mathbb{R},A)$-Lie algebra of smooth vector fields on W the exact sequence (6.5.1) boils down to the exact sequence

$$0\to \mathrm{H}^1(W,\mathbb{R}^\times_+)\to \mathrm{H}^1(W,\mathbb{R}^\times)\to \mathrm{H}^1(W,\mathbb{Z}/2)\to 0$$

associated with the split exact coefficient sequence

$$0\to \mathbb{R}^\times_+\to \mathbb{R}^\times\to \mathbb{Z}/2\to 0$$

of abelian groups. For a general $(\mathbb{R},A)$-Lie algebra L of the kind coming into play in (6.7), by naturality, the corresponding morphism $L\to \mathrm{Vect}(W)$ of $(\mathbb{R},A)$-Lie algebras induces a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{H}^1(W,\mathbb{R}^\times_+) & \longrightarrow & \mathrm{H}^1(W,\mathbb{R}^\times) & \longrightarrow & \mathrm{H}^1(W,\mathbb{Z}/2) & \longrightarrow & 0 \\ & & \Big\downarrow & & \Big\downarrow & & \Big\downarrow{\cong} & & \\ 0 & \longrightarrow & \mathcal{H}^1(L,A) & \longrightarrow & \mathrm{Pic}^{\mathrm{flat}}(L,A) & \longrightarrow & \mathrm{Pic}(A) & \longrightarrow & 0 \end{array} \tag{6.8.1}$$

of abelian groups. Thus the group $\mathrm{Pic}^{\mathrm{flat}}(L,A)$ may be constructed from $\mathrm{H}^1(W,\mathbb{R}^\times)$ and the group $\mathcal{H}^1(L,A)$ of line bundles with a flat L-connection which are trivial as line bundles.

Remark 6.9. Still under the circumstances of (6.7), when A is the algebra of smooth complex functions on W and L the $(\mathbb{C},A)$-Lie algebra arising from complexification of the Lie algebra of smooth vector fields on W, the exact sequence (6.5.1) boils down to the exact sequence

$$0\to \mathrm{H}^1(W,\mathbb{C}^\times)_0\to \mathrm{H}^1(W,\mathbb{C}^\times)\to \mathrm{H}^2(W,\mathbb{Z})\to \mathrm{H}^2(W,\mathbb{C});$$

here $\mathrm{H}^1(W,\mathbb{C}^\times)\cong \mathrm{Hom}(\pi_1(W),\mathbb{C}^\times)$, the group of flat complex line bundles and $\mathrm{H}^1(W,\mathbb{C}^\times)_0$ is the group of flat complex line bundles which are topologically trivial; this group may also be described as the connected component of the trivial homomorphism in $\mathrm{Hom}(\pi_1(W),\mathbb{C}^\times)$ or as the quotient $\mathrm{H}^1(W,\mathbb{C})\big/\mathrm{H}^1(W,\mathbb{Z})$. Notice

that the second cohomology group $\mathrm{H}^2(W, \mathbb{Z})$ with integer coefficients amounts to the Picard group of the ring A of smooth complex functions on W.

EXAMPLE 6.10. Under the circumstances of (6.7), suppose that L arises from the tangent bundle of a foliation $\mathcal{F}$ on W. Then the line bundle which corresponds to Q_L amounts to the top exterior power of the conormal bundle to $\mathcal{F}$. The left (A, L)-module structure on Q_L is then the *Bott connection* and, provided the normal bundle of $\mathcal{F}$ is orientable, the elements of Q_L are transverse measures to $\mathcal{F}$. With the definition of modular class given in [10], the modular class of Q_L is there called the *modular class of the foliation*. See also [22].

EXAMPLE 6.11. Let $L = \mathfrak{g}$ be an ordinary Lie algebra of finite dimension n (say) over a field k, considered as the Lie-Rinehart algebra $(k, \mathfrak{g})$, with $A = k$. Then the group $\mathcal{H}^1(L, A)$ boils down to $\mathrm{H}^1(\mathfrak{g}, k) = \mathrm{Hom}(\mathfrak{g}, k)$, and the exact sequence (6.5.1) yields an isomorphism from $\mathrm{H}^1(\mathfrak{g}, k)$ onto $\mathrm{Pic}^{\mathrm{flat}}(L, A)$. Moreover, $C_{\mathfrak{g}} = \Lambda^n_R \mathfrak{g}^*$ and $Q_{\mathfrak{g}} = \mathrm{Hom}_k(C_{\mathfrak{g}}, k) = \Lambda^n_R \mathfrak{g}$, with the obvious respective right- and left $\mathfrak{g}$-module structures, cf. (1.5) and (4.5). Thus, as an element of $\mathrm{H}^1(\mathfrak{g}, k)$, the modular class then comes down to the adjoint character $\xi_0 \colon \mathfrak{g} \to k$ given by

$$\xi_0(x) = \mathrm{tr}(\mathrm{ad}_x), \quad x \in \mathfrak{g}.$$

Given a general Lie-Rinehart algebra (A, L) over a commutative ring R and a free A-module M of rank one and being endowed with a left (A, L)-module structure, with reference to a basis element b of M, for $\alpha \in L$, we define the *divergence* $\mathrm{div}_b \alpha$ of α by

$$\alpha_M(b) = (\mathrm{div}_b \alpha) b. \tag{6.12}$$

EXAMPLE 6.13. Let $L = A \otimes_R \mathfrak{g}$ be an (R, A)-Lie algebra of the kind considered in (2.12). In view of (2.12), the dualizing module C_L of L may be written $C_L = A \otimes_R \Lambda^n \mathfrak{g}^*$ and, in this description, the right (A, L)-module structure on C_L is given by (2.12.1). Hence, as an A-module, $Q_L = \mathrm{Hom}_A(C_L, \omega_A)$ is plainly isomorphic to

$$\mathrm{Hom}_R(C_{\mathfrak{g}}, \omega_A) \cong \Lambda^n \mathfrak{g} \otimes_R \omega_A.$$

Now the A-dual $\mathrm{Hom}_A(C_L, A)$ of the dualizing module C_L manifestly has the form $A \otimes_R \Lambda^n \mathfrak{g}$ and from this description it is obvious that (2.12.4) turns this A-dual $\mathrm{Hom}_A(C_L, A)$ into a left (A, L)-module; likewise ω_A inherits a left (A, L)-module structure and, with reference to these structures, the canonical A-module isomorphism

$$\mathrm{Hom}_A(C_L, \omega_A) \cong (A \otimes_R \Lambda^n \mathfrak{g}) \otimes_A \omega_A \tag{6.13.1}$$

is one of left (A, L)-modules, the left (A, L)-module structure on the right-hand side being given by (2.1). This description of $Q_L = \mathrm{Hom}_A(C_L, \omega_A)$ is somewhat simpler than that given in (6.6) above for a *general* (R, A)-Lie algebra. If, moreover, as an A-module, ω_A is free, with basis b (say), the modular class of $Q_L \cong \Lambda^n \mathfrak{g} \otimes_R \omega_A$ lies in the subgroup $\mathcal{H}^1(L, A)$ of $\mathrm{Pic}^{\mathrm{flat}}(L, A)$ and is the class of

$$\xi_0 + \mathrm{div}_b \in \mathrm{Der}(\mathfrak{g}, A) \cong \mathrm{Der}(L, A) \tag{6.13.2}$$

ξ_0 being the adjoint character given in (6.11). This follows from the description (6.13.1) of the left (A, L)-module structure on Q_L for the special case under consideration. See also what is said in [10].

7. Poisson algebras

Let A be a Poisson algebra, with Poisson structure $\{\cdot,\cdot\}$, and let $D_{\{\cdot,\cdot\}}$ be its A-module of formal differentials D_A or a suitable quotient $\overline{D}_A$ thereof so that the canonical map from $\mathrm{Hom}_A(\overline{D}_A, A)$ to $\mathrm{Hom}_A(D_A, A)(\cong \mathrm{Der}(A))$ is an isomorphism. Suppose that $D_{\{\cdot,\cdot\}}$ is endowed with the (R, A)-Lie algebra structure induced by the Poisson structure; see [15] (3.8) for details. We further suppose that, as an A-module, D_A or the appropriate quotient $\overline{D}_A$ thereof is finitely generated and projective of constant rank n; then the A-module $\mathrm{Der}(A)$ is finitely generated and projective of constant rank n as well, and A is regular in a sense explained in Section 3 above. For example, when A is the algebra of smooth functions on a smooth real Poisson manifold P of dimension n, the appropriate object to be taken is the A-module of 1-forms $\overline{D}_A = \Omega^1(P)$, with its $(\mathbb{R}, A)$-Lie algebra structure. For a general Poisson algebra A of the kind under discussion, given left- and right $(A, D_{\{\cdot,\cdot\}})$-modules M and N, *Poisson cohomology* $\mathrm{H}^*_{\mathrm{Poisson}}(A, M)$ and *Poisson homology* $\mathrm{H}_*^{\mathrm{Poisson}}(A, N)$ are then given by (or, indeed, may then be defined by)

$$\mathrm{H}^*_{\mathrm{Poisson}}(A, M) = \mathrm{H}^*(D_{\{\cdot,\cdot\}}, M), \quad \mathrm{H}_*^{\mathrm{Poisson}}(A, N) = \mathrm{H}_*(D_{\{\cdot,\cdot\}}, N).$$

(When A is not regular in our sense, this may not be the appropriate definition; see [15] for details.) The dualizing module for $D_{\{\cdot,\cdot\}}$, which we denote by $C_{\{\cdot,\cdot\}}$, is just

$$C_{\{\cdot,\cdot\}} = \Lambda^n_A D^*_{\{\cdot,\cdot\}} = \mathrm{Hom}_A(\Lambda^n_A D_{\{\cdot,\cdot\}}, A) \cong \Lambda^n_A \mathrm{Der}(A), \tag{7.1}$$

the last isomorphism being one of A-modules, so that the right $(A, D_{\{\cdot,\cdot\}})$-module structure on $\Lambda^n_A\mathrm{Der}(A)$ is induced by that on $\mathrm{Hom}_A(\Lambda^n_A D_{\{\cdot,\cdot\}}, A)$ via this isomorphism. Given a right $(A, D_{\{\cdot,\cdot\}})$-module N, the inverse duality isomorphism (2.11.2) now manifestly has the form

$$\mathrm{H}_k^{\mathrm{Poisson}}(A, N) \longrightarrow \mathrm{H}^{n-k}_{\mathrm{Poisson}}(A, \mathrm{Hom}_A(C_{\{\cdot,\cdot\}}, N)), \tag{7.2}$$

for every non-negative integer k. By Theorem 3.7, (7.2) may be obtained as the cap product with a suitable fundamental class. Moreover, the proof of Proposition 1.4 shows that an isomorphism of the kind (7.2) may even be taken to be induced by an isomorphism of complexes: Let $K = K(A, D_{\{\cdot,\cdot\}})$, the Rinehart complex reproduced in (2.9.2), which is in fact a projective resolution of A in the category of left $(A, D_{\{\cdot,\cdot\}})$-modules, since $D_{\{\cdot,\cdot\}}$ is assumed to be projective as an A-module; with this choice of K, the left-hand side of the isomorphism (1.4.2) is the ordinary complex calculating Poisson homology with coefficients in N while the right-hand side calculates the corresponding Poisson cohomology with coefficients in $\mathrm{Hom}_A(C_{\{\cdot,\cdot\}}, N)$, cf. [15]. In particular, besides its obvious left $(A, D_{\{\cdot,\cdot\}})$-module structure, the algebra A inherits also a *right* $(A, D_{\{\cdot,\cdot\}})$-module structure; it is given the formula

$$a(b(du)) = \{ab, u\}, \quad a, b, u \in A, \tag{7.3}$$

where $du \in D_A$ refers to the formal differential of $u \in A$. We denote the resulting right $(A, D_{\{\cdot,\cdot\}})$-module by $A_{\{\cdot,\cdot\}}$. As a special case of (7.2), with $N = A_{\{\cdot,\cdot\}}$, we now obtain the isomorphism

$$\mathrm{H}_k^{\mathrm{Poisson}}(A, A_{\{\cdot,\cdot\}}) \to \mathrm{H}^{n-k}_{\mathrm{Poisson}}(A, \mathrm{Hom}_A(C_{\{\cdot,\cdot\}}, A_{\{\cdot,\cdot\}})). \tag{7.4}$$

Here $\mathrm{Hom}_A(C_{\{\cdot,\cdot\}}, A_{\{\cdot,\cdot\}})$ is endowed with the left $(A, D_{\{\cdot,\cdot\}})$-module structure given by (2.3). We note that $\mathrm{Hom}_A(C_{\{\cdot,\cdot\}}, A_{\{\cdot,\cdot\}})$ is plainly isomorphic to $\Lambda^n_A D_{\{\cdot,\cdot\}}$ as an A-module. Further, in view of Proposition 4.6 (ii), $\mathrm{Hom}_A(C_{\{\cdot,\cdot\}}, A_{\{\cdot,\cdot\}})$ looks like a trace module, with $V = A_{\{\cdot,\cdot\}}$. We shall come back to this observation later. When A is the Poisson algebra of smooth functions on a smooth real Poisson manifold P and $D_{\{\cdot,\cdot\}}$ the A-module of 1-forms $\Omega^1(P)$, with its $(\mathbb{R}, A)$-Lie algebra structure mentioned at the beginning of this section, the isomorphism (7.4) is precisely the one given in [10] and written

$$\mathrm{H}^k(P, \Lambda^{\mathrm{top}}\mathrm{T}^*P) \cong \mathrm{H}_{n-k}(P)$$

there. A special case thereof may be found in Section 6 of [8].

Next we consider the left $(A, D_{\{\cdot,\cdot\}})$-module $Q_{D_{\{\cdot,\cdot\}}}$, cf. (6.6) for its description. Under the present circumstances, by definition, this module is of the form

$$Q_{D_{\{\cdot,\cdot\}}} = \mathrm{Hom}_A(C_{\{\cdot,\cdot\}}, \omega_A) \tag{7.5.1}$$

where the left $(A, D_{\{\cdot,\cdot\}})$-module structure on $\mathrm{Hom}_A(C_{\{\cdot,\cdot\}}, \omega_A)$ is given by (2.3), with reference to the right $(A, D_{\{\cdot,\cdot\}})$-module structures on $C_{\{\cdot,\cdot\}}$ and ω_A. On the other hand, as A-modules,

$$\mathrm{Hom}_A(C_{\{\cdot,\cdot\}}, A) \otimes_A \omega_A \cong \Lambda^n_A D_{\{\cdot,\cdot\}} \otimes_A \omega_A, \tag{7.5.2}$$

and the corresponding left $(A, D_{\{\cdot,\cdot\}})$-module structure on $\Lambda^n_A D_{\{\cdot,\cdot\}} \otimes_A \omega_A$ may be described as being given by the Lie derivative on both $\Lambda_A D_{\{\cdot,\cdot\}}$ and

$$\omega_A = \mathrm{Hom}_A(\Lambda^n_A \mathrm{Der}(A), A) = \Lambda^n_A D_A. \tag{7.6}$$

We shall say a bit more about this description in the proof of Theorem 7.9 below. However, as A-modules,

$$\Lambda^n_A D_{\{\cdot,\cdot\}} \cong \mathrm{Hom}_A(C_{\{\cdot,\cdot\}}, A), \quad \omega_A \cong \mathrm{Hom}_A(C_{\{\cdot,\cdot\}}, A) \tag{7.7}$$

and, as we have already observed, with reference to the right $(A, D_{\{\cdot,\cdot\}})$-module structure on $C_{\{\cdot,\cdot\}}$ and that on A given by (7.3) above (indicated by the notation $A_{\{\cdot,\cdot\}}$), (2.3) endows $\mathrm{Hom}_A(C_{\{\cdot,\cdot\}}, A_{\{\cdot,\cdot\}})$ with a left $(A, D_{\{\cdot,\cdot\}})$-module structure.

Lemma 7.8. *This left $(A, D_{\{\cdot,\cdot\}})$-module structure on $\mathrm{Hom}_A(C_{\{\cdot,\cdot\}}, A_{\{\cdot,\cdot\}}) \cong \Lambda^n_A D_{\{\cdot,\cdot\}}$ is given by the formula*

$$(du)(\beta) = \lambda_{du}(\beta), \quad u \in A, \ \beta \in \Lambda^n_A D_{\{\cdot,\cdot\}}.$$

Here "λ" refers to the operation of Lie derivative (2.6.2); notice that we do *not* assert that the left $(A, D_{\{\cdot,\cdot\}})$-module structure on $\mathrm{Hom}_A(C_{\{\cdot,\cdot\}}, A_{\{\cdot,\cdot\}})$ is given by

the operation of Lie derivative on $\Lambda_A^n D_{\{\cdot,\cdot\}}$; only the result of the operation with the A-module *generators* du ($u \in A$) of $D_{\{\cdot,\cdot\}}$ is given in this way. We note that, when $\Lambda_A^n D_{\{\cdot,\cdot\}}$ is actually free (necessarily of rank 1) as an A-module, a choice of basis element β of $\Lambda_A^n D_{\{\cdot,\cdot\}}$ determines a derivation Φ_β of A such that, for $u \in A$, the result $\lambda_{du}(\beta)$ is given by the formula

$$\lambda_{du}(\beta) = (\Phi_\beta u)\beta.$$

When A is the algebra of smooth functions on a smooth (orientable) Poisson manifold P, this derivation Φ_β is precisely the "modular vector field" or "curl" ("rotationnel" in French), that is, the infinitesimal generator of the modular automorphism group of P; this automorphism group was introduced in [22], and the modular vector field is written there in the form $\mathcal{L}_{H_f}\mu/\mu$. The modular vector field occurs already in [18]; see [22] for its significance and for additional references.

Proof. Write $\Lambda = \Lambda_A^n D_{\{\cdot,\cdot\}}$. Then $C_{\{\cdot,\cdot\}} \cong \mathrm{Hom}_A(\Lambda, A)$ as A-modules and, for $\beta \in \Lambda$ and $\phi \in \mathrm{Hom}_A(\Lambda, A)$, the assignment $\beta(\phi) = \phi(\beta)$ identifies Λ with

$$\mathrm{Hom}_A(C_{\{\cdot,\cdot\}}, A) \cong \mathrm{Hom}_A(\mathrm{Hom}_A(\Lambda, A), A)$$

as A-modules. Let $\alpha \in D_{\{\cdot,\cdot\}}$. Then

$$\begin{aligned}(\alpha\beta)(\phi) &= \beta(\phi\alpha) - (\beta\phi)\alpha \\ &= (\phi\alpha)(\beta) - (\phi(\beta))\alpha \\ &= \phi(\lambda_\alpha(\beta)) - \alpha(\phi(\beta)) - (\phi(\beta))\alpha,\end{aligned}$$

that is to say,

$$\phi(\alpha(\beta)) = \phi(\lambda_\alpha(\beta)) - \alpha(\phi(\beta)) - (\phi(\beta))\alpha.$$

Let $u \in A$ and $\alpha = du$; we then have

$$\alpha(\phi(\beta)) = \{u, \phi(\beta)\}, \quad (\phi(\beta))\alpha = \{\phi(\beta), u\},$$

whence $\phi(\alpha(\beta)) = \phi(\lambda_\alpha(\beta))$. Since ϕ is arbitrary, we conclude that $\alpha(\beta) = \lambda_\alpha(\beta)$. This proves the assertion. $\square$

As a consequence we obtain the following.

Theorem 7.9. *With reference to the right $(A, D_{\{\cdot,\cdot\}})$-module $A_{\{\cdot,\cdot\}}$ and the left $(A, D_{\{\cdot,\cdot\}})$-module structure on $\mathrm{Hom}_A(C_{\{\cdot,\cdot\}}, A_{\{\cdot,\cdot\}})$ (given by (2.3)), we have*

$$Q_{D_{\{\cdot,\cdot\}}} \cong \mathrm{Hom}_A(C_{\{\cdot,\cdot\}}, A_{\{\cdot,\cdot\}}) \otimes_A \mathrm{Hom}_A(C_{\{\cdot,\cdot\}}, A_{\{\cdot,\cdot\}}) \tag{7.9.1}$$

as left $(A, D_{\{\cdot,\cdot\}})$-modules.

Proof. For a general duality (R, A)-Lie algebra L of rank n, it is readily seen that, when the left (A, L)-module $Q_L = \mathrm{Hom}_A(C_L, \omega_A)$ (cf. (6.6)) is written in the form

$$Q_L = \Lambda_A^n L \otimes_A \omega_A,$$

the left L-module structure underlying the left (A, L)-module structure of Q_L may be described as the ordinary tensor product L-action arising from the operations of Lie derivative on the tensor factors $\Lambda^n_A L$ and ω_A; neither of these Lie derivative actions defines a left (A, L)-module structure separately but their combination indeed yields such a structure on the tensor product. This remark applies in particular to $L = D_{\{\cdot,\cdot\}}$ and $\Lambda^n_A D_{\{\cdot,\cdot\}} \otimes_A \omega_A$. By Lemma 7.8, the result of the operation on $\mathrm{Hom}_A(C_{\{\cdot,\cdot\}}, A_{\{\cdot,\cdot\}})$ with the A-module generators du $(u \in A)$ of $D_{\{\cdot,\cdot\}}$ is given by the operation of Lie derivative. In view of the axioms for a general (R, A)-Lie algebra L and of those for a left (A, L)-module, the left $(A, D_{\{\cdot,\cdot\}})$-module structure on $\mathrm{Hom}_A(C_{\{\cdot,\cdot\}}, A_{\{\cdot,\cdot\}})$ is then completely determined, and the claim follows. $\square$

Remark 7.10. Theorem 7.9 generalizes a corresponding result in [10], and our argument involving the right $(A, D_{\{\cdot,\cdot\}})$-module $A_{\{\cdot,\cdot\}}$ simplifies the proof thereof. This result has also been generalized in [23], where other issues related to the present paper are discussed, too. See as well our follow up paper [16].

We now return to our general Poisson algebra A and write

$$Q_{\{\cdot,\cdot\}} = \mathrm{Hom}_A(C_{\{\cdot,\cdot\}}, A_{\{\cdot,\cdot\}});$$

the class $[Q_{\{\cdot,\cdot\}}]$ in $\mathrm{Pic}^{\mathrm{flat}}(D_{\{\cdot,\cdot\}}, A)$ of $Q_{\{\cdot,\cdot\}}$ is plainly characteristic for the Poisson algebra A, and we refer to it as the *modular class* of A. In view of Theorem 7.9,

$$[Q_{\{\cdot,\cdot\}}]^2 = [Q_{D_{\{\cdot,\cdot\}}}] \in \mathrm{Pic}^{\mathrm{flat}}(D_{\{\cdot,\cdot\}}, A),$$

that is, the modular class of the Lie-Rinehart algebra $(A, D_{\{\cdot,\cdot\}})$ is twice the modular class (or its square when we think of the group structure as being multiplicative) of the Poisson algebra A. We note that, when A is the Poisson algebra of smooth real functions on a smooth compact Poisson manifold P, the modular class is zero if and only if P has a global volume form which is annihilated by all Hamiltonian vector fields.

Remark 7.11. When A is the Poisson algebra of smooth real functions on a smooth Poisson manifold P, our definition of the modular class of A is related to the definition of the modular class of P in [22] in the same way as our definition of the modular class $[Q_L]$ for a general Lie-Rinehart algebra (A, L) is related to the definition of the modular class of a Lie algebroid given in [10], cf. Remark 6.7 above.

Illustration 7.12. Let A be a Poisson algebra of functions on the dual of an R-Lie algebra $\mathfrak{g}$, the Poisson bracket being induced by the Lie bracket on $\mathfrak{g}$ in the customary way; so A could be the algebra of smooth functions on the dual $\mathfrak{g}^*$ of an ordinary real (or complex) Lie algebra $\mathfrak{g}$, or A could be the algebra of polynomials on an R-Lie algebra $\mathfrak{g}$ for a general ground ring R. We suppose that, as an R-module, $\mathfrak{g}$ is finitely generated and projective of constant rank n (say); when R is a field this is of course just the dimension of $\mathfrak{g}$. Now $\mathfrak{g}$ acts on A in an obvious fashion in such a way that $D_{\{\cdot,\cdot\}}$ is isomorphic to the corresponding (R, A)-Lie algebra $A \otimes_R \mathfrak{g}$ explained in (2.12) above. See also [15] (3.18). This implies that $C_{\{\cdot,\cdot\}}$ is isomorphic to $A \otimes_R \Lambda^n \mathfrak{g}^*$ whence $\mathrm{Hom}_A(C_{\{\cdot,\cdot\}}, A)$ is isomorphic to $A \otimes_R \Lambda^n \mathfrak{g}$, the $\mathfrak{g}$-structures being the obvious ones. Moreover, $Q_{\{\cdot,\cdot\}}$ is isomorphic to $A \otimes_R \Lambda^n \mathfrak{g} \otimes_R \Lambda^n \mathfrak{g}$, and the isomorphism (7.9.1) now comes down to the obvious isomorphism between $A \otimes_R \Lambda^n \mathfrak{g} \otimes_R \Lambda^n \mathfrak{g}$ and $(A \otimes_R \Lambda^n \mathfrak{g}) \otimes_A (A \otimes_R \Lambda^n \mathfrak{g})$.

Corollary 7.13. *The pairing* (3.11) *induces a bilinear pairing*

$$\mathrm{H}_k^{\mathrm{Poisson}}(A, A_{\{\cdot,\cdot\}}) \otimes_{\mathbb{R}} \mathrm{H}_{n-k}^{\mathrm{Poisson}}(A, A_{\{\cdot,\cdot\}}) \longrightarrow \omega_A \otimes_U A. \tag{7.13.1}$$

Proof. The cohomology pairing (3.9) with reference to the resulting pairing

$$\mathrm{Hom}_A(C_{\{\cdot,\cdot\}}, A_{\{\cdot,\cdot\}}) \otimes_A \mathrm{Hom}_A(C_{\{\cdot,\cdot\}}, A_{\{\cdot,\cdot\}}) \longrightarrow Q_{D_{\{\cdot,\cdot\}}} = \mathrm{Hom}_A(C_{\{\cdot,\cdot\}}, \omega_A)$$

of left $(A, D_{\{\cdot,\cdot\}})$-modules takes the form

$$\begin{gathered}\mathrm{H}^k_{\mathrm{Poisson}}(A, \mathrm{Hom}_A(C_{\{\cdot,\cdot\}}, A_{\{\cdot,\cdot\}})) \otimes_R \mathrm{H}^{n-k}_{\mathrm{Poisson}}(A, \mathrm{Hom}_A(C_{\{\cdot,\cdot\}}, A_{\{\cdot,\cdot\}})) \\ \longrightarrow C_{\{\cdot,\cdot\}} \otimes_U \mathrm{Hom}_A(C_{\{\cdot,\cdot\}}, \omega_A) \cong \omega_A \otimes_U A\end{gathered}$$

where $U = U(A, D_{\{\cdot,\cdot\}})$ denotes the corresponding universal algebra. Naive duality (cf. (2.11) above) carries this pairing into (7.13.1). □

When A is the Poisson algebra of smooth real functions on an orientable Poisson manifold P, integration yields a map from $\omega_A \otimes_U A$ to the reals which is non-zero when the manifold is compact or when we are working with compactly supported functions and forms and hence we then obtain a bilinear real-valued pairing

$$\mathrm{H}_k^{\mathrm{Poisson}}(P) \otimes_{\mathbb{R}} \mathrm{H}_{n-k}^{\mathrm{Poisson}}(P) \longrightarrow \mathbb{R}. \tag{7.14}$$

This is precisely the pairing given in of [10]. For reasons similar to those explained in (3.10), this pairing will not be nondegenerate unless the Poisson structure is a symplectic one. In the light of the notion of Poincaré duality introduced in Section 4, we now point a way towards truely nondegenerate (co)homology pairings in Poisson (co)homology.

Example 7.15. Let A be the algebra of smooth real functions on a smooth n-dimensional manifold B, endowed with the trivial Poisson structure. The corresponding $(\mathbb{R}, A)$-Lie algebra is just the space D of sections of the cotangent bundle of B with trivial Lie bracket, and the Poisson cohomology $\mathrm{H}^*_{\mathrm{Poisson}}(A, A)$ is the graded A-algebra $\Lambda_A V$ of multi vector fields on B, where we have written $V = \mathrm{Vect}(B)$, the Lie bracket on V being ignored. In particular, letting $\mathcal{O} = \Lambda_A^n D$, we have

$$\mathrm{H}^n_{\mathrm{Poisson}}(A, \mathcal{O}) = \mathrm{H}^n(D, \mathcal{O}) = \mathrm{Hom}_A(\Lambda_A^n D, \Lambda_A^n D) = A,$$

and this defines a trace $t \colon \mathrm{H}^n(D, \mathcal{O}) \to A$. Poincaré duality now comes down to the fact that the canonical pairing

$$\Lambda_A^* D \otimes_A \Lambda_A^{n-*} D \longrightarrow \Lambda_A^* D$$

is a perfect pairing of A-modules. In general, this notion of duality cannot even be phrased as one of real vector spaces! Thus to understand Poincaré duality for Lie-Rinehart algebras one is forced to admit ground rings more general than the naive ones.

We now switch to non-trivial Poisson structures. Thus let $\{\cdot,\cdot\}$ be a non-trivial Poisson structure on A. In view of what has been said in Section 4, see

in particular (4.15), the appropriate ground ring to be taken is the subalgebra $R = \mathrm{H}^0_{\mathrm{Poisson}}(A, A) \subseteq A$ of Casimir elements in A. Under suitable circumstances, a trace can now be constructed by means of (4.6) (ii), with $V = A_{\{\cdot,\cdot\}}$ (with the right $(A, D_{\{\cdot,\cdot\}})$-module structure (7.3)): The isomorphism ι (cf. (4.6)) then comes down to an isomorphism

$$\iota\colon A_{\{\cdot,\cdot\}} \otimes_U A \cong \mathrm{H}_0^{\mathrm{Poisson}}(A, A) \xrightarrow{\iota} \mathrm{H}^0_{\mathrm{Poisson}}(A, A) = R,$$

that is, a trace boils essentially down to an isomorphism from $A_{\{\cdot,\cdot\}} \otimes_U A$ onto the algebra of Casimir elements such that, for every right duality $(A, D_{\{\cdot,\cdot\}})$-module N, the canonical map (4.6.3) with $L = D_{\{\cdot,\cdot\}}$ is an isomorphism. We now explain a situation where indeed a trace is obtained in this way and where Poincaré duality holds, with the algebra of Casimir elements as ground ring.

Example 7.16. Let W be a smooth real n-dimensional Poisson manifold, A its Poisson algebra of smooth functions, and suppose that the symplectic foliation $\mathcal{F}$ of W constitutes a fiber bundle with compact fiber F, of dimension m, and base B. For example, W could be the total space of a fiber bundle whose fiber has a symplectic structure preserved by the structure group. An example in which the cohomology class of the symplectic structure "varies" from fiber to fiber arises from the regular part of the dual of a semisimple Lie algebra of compact type: the fibers are flag manifolds with a fixed complex structure but with all "different" symplectic structures; this follows from results in [4]. I am indebted to A. Weinstein for having pointed out these examples to me. The algebra $\mathrm{H}^0_{\mathrm{Poisson}}(A, A)$ of Casimir elements now amounts to the algebra $C^\infty(B)$ of smooth functions on B. Further, the image of the structure map from $D_{\{\cdot,\cdot\}}$ to $\mathrm{Vect}(W)$ is the $(\mathbb{R}, A)$-Lie algebra $L_{\mathcal{F}}$ which arises from the Lie algebroid giving the infinitesimal structure of the foliation $\mathcal{F}$, and the resulting surjection from $D_{\{\cdot,\cdot\}}$ to $L_{\mathcal{F}}$ fits into an extension

$$0 \to L' \to D_{\{\cdot,\cdot\}} \to L_{\mathcal{F}} \to 0 \tag{7.16.1}$$

of $(\mathbb{R}, A)$-Lie algebras which, in turn, comes from an extension of Lie algebroids; in particular, besides $D_{\{\cdot,\cdot\}}$ which has as underlying vector bundle the cotangent bundle of W, the $(\mathbb{R}, A)$-Lie algebras L' and $L_{\mathcal{F}}$ have underlying vector bundles, too. Let

$$\tau\colon \mathrm{H}^k(L_{\mathcal{F}}, \mathcal{O}_{\mathcal{F}}) \xrightarrow{e\cap\,\cdot} C_{L_{\mathcal{F}}} \otimes_U \mathcal{O}_{\mathcal{F}} \to C^\infty(B)$$

be the corresponding trace (4.15.1) for $L_{\mathcal{F}}$, and let $(\mathcal{O}, t)$ be the resulting trace for $D_{\{\cdot,\cdot\}}$ given by (5.2). We record the isomorphisms

$$\mathcal{O} = \mathrm{Hom}_A(C_{\{\cdot,\cdot\}}, C_{\mathcal{F}}) \otimes_A \mathcal{O}_{\mathcal{F}} \cong \mathrm{Hom}_A(C_{\{\cdot,\cdot\}}, C_{\mathcal{F}} \otimes_A \mathcal{O}_{\mathcal{F}}) \cong \mathrm{Hom}_A(C_{\{\cdot,\cdot\}}, A_{\{\cdot,\cdot\}})$$

of left $(A, D_{\{\cdot,\cdot\}})$-modules where $\mathrm{Hom}_A(C_{\{\cdot,\cdot\}}, A_{\{\cdot,\cdot\}})$ is endowed with the left $(A, D_{\{\cdot,\cdot\}})$-module structure (2.3); now, as an A-module, $\mathrm{Hom}_A(C_{\{\cdot,\cdot\}}, A_{\{\cdot,\cdot\}})$ is isomorphic to $\Lambda^n_A D_{\{\cdot,\cdot\}}$, that is, to the space of sections of the highest non-zero exterior power of the cotangent bundle of W. Thus, as an A-module, $\mathcal{O}$ is just a copy of A if and only if W is orientable.

Theorem 7.16.2. *Under the present circumstances, the Poisson cohomology cup pairing*

$$\mathrm{H}^k_{\mathrm{Poisson}}(A, A) \otimes_{C^\infty(B)} \mathrm{H}^{n-k}_{\mathrm{Poisson}}(A, \mathcal{O}) \longrightarrow \mathrm{H}^n_{\mathrm{Poisson}}(A, \mathcal{O}) \xrightarrow{t} C^\infty(B) \tag{7.16.3}$$

is a nondegenerate bilinear pairing of finitely generated projective $(C^\infty(B))$*-modules.*

In view of the naturality of the duality isomorphism (3.7.1), naive duality (cf. (2.11) above) carries the pairing (7.16.3) into a bilinear pairing in Poisson homology, and we obtain at once the following.

Corollary 7.16.4. *Under the circumstances of* (7.16.2)*, the pairing* (7.16.3) *induces the nondegenerate bilinear pairing*

$$\mathrm{H}_{n-k}^{\mathrm{Poisson}}(A, C_{\{\cdot,\cdot\}}) \otimes_{C^\infty(B)} \mathrm{H}_k^{\mathrm{Poisson}}(A, A_{\{\cdot,\cdot\}}) \longrightarrow C^\infty(B) \tag{7.16.5}$$

of finitely generated projective $(C^\infty(B))$*-modules.* $\square$

Thus, in Poisson (co)homology, pairings of the kind (7.16.3) and (7.16.5) are more appropriate objects of study than those of the kind (7.14), which are rarely nondegenerate, as already pointed out. Furthermore, the pairings (7.16.3) and (7.16.5) do not pose any problem with the interpretation of the term "nondegenerate" since the modules involved are finitely generated over the corresponding ground ring, the subalgebra of Casimir functions (but in general not finitely generated over the reals).

We now prepare for the proof of Theorem 7.16.2. To elucidate the structure of the extension (7.16.1), consider the extension

$$0 \longrightarrow \kappa_F \longrightarrow \tau_W \longrightarrow W \times_B \tau_B \longrightarrow 0 \tag{7.16.6}$$

of vector bundles on W arising from the projection of the tangent bundle τ_W of W to the tangent bundle τ_B of B. The induced extension of the dual bundles may be written

$$0 \longrightarrow W \times_B \tau_B^* \longrightarrow \tau_W^* \longrightarrow \kappa_F^* \longrightarrow 0. \tag{7.16.7}$$

The spaces of sections of this extension yield the extension of A-modules which underlies (7.16.1); in particular, we deduce that L' arises from the projective $(C^\infty(B))$-module $\Gamma(\tau_B^*)$ by extension of scalars, that is, $L' \cong A \otimes_{C^\infty(B)} \Gamma(\tau_B^*)$, and this is an isomorphism of A-Lie algebras, when $\Gamma(\tau_B^*)$ and hence $A \otimes_{C^\infty(B)} \Gamma(\tau_B^*)$ are endowed with the trivial Lie bracket. Now $\mathrm{H}^*(L', A)$ is just the exterior A-algebra of L' and, since L' is an induced module,

$$\mathrm{H}^*(L', A) \cong A \otimes_{C^\infty(B)} \left(\Lambda^*_{C^\infty(B)} \Gamma(\tau_B^*)\right). \tag{7.16.8}$$

Recall from (2.10) that the dualizing module C_L for $L = D_{\{\cdot,\cdot\}}$ has the form $C_L = \mathrm{Hom}_A(\Lambda^n_A L, A)$, endowed with the right (A, L)-module structure (2.8.1), for $M = A$. Inspection shows that, under the present circumstances, the operation of Lie derivative of L on C_L factors through the quotient $L_{\mathcal{F}}$, and hence the right

(A,L)-module structure on C_L factors through a right $(A,L_{\mathcal F})$-module structure. Consequently,

$$\mathrm{H}_*(L',C_L) \cong \left(\Lambda^*_{C^\infty(B)}\Gamma(\tau^*_B)\right) \otimes_{C^\infty(B)} C_L \tag{7.16.9}$$

as right $(A,L_{\mathcal F})$-modules, the right-hand side being endowed with the obvious right $(A,L_{\mathcal F})$-module structure coming from that on C_L.

Proof of Theorem 7.16.2. As a $(C^\infty(B))$-module, each module $E_r^{p,q}(A)$ occurring in the cohomology spectral sequence $(E_r^{p,q}(A),d_r)$ for the extension (7.16.1) with coefficients in A is projective. Hence each module $E_\infty^{p,q}(A)$ is projective as a $(C^\infty(B))$-module, whence so are the cohomology modules $\mathrm{H}^*_{\mathrm{Poisson}}(A,A)$ and $\mathrm{H}^*_{\mathrm{Poisson}}(A,\mathcal O)$.

By Corollary 4.15.7, the $(\mathbb R,A)$-Lie algebra $L_{\mathcal F}$, endowed with the trace $(\mathcal O_{\mathcal F},\tau)$, satisfies Poincaré duality for every finitely generated projective induced left (A,L)-module of the kind $M = A\otimes_{C^\infty(B)} M'$, M' denoting an arbitrary finitely generated projective $(C^\infty(B))$-module. Consequently $L_{\mathcal F}$ satisfies Poincaré duality for the right $(A,L_{\mathcal F})$-modules

$$\mathrm{H}_0(L',C_L),\mathrm{H}_1(L',C_L),\dots,\mathrm{H}_k(L',C_L).$$

We now apply Theorem 5.3 and deduce that the $(\mathbb R,A)$-Lie algebra $D_{\{\cdot,\cdot\}}$, endowed with the trace $(\mathcal O,t)$, satisfies Poincaré duality for the right (A,L)-module C_L. Consequently $D_{\{\cdot,\cdot\}}$ satisfies Poincaré duality for the left (A,L)-module $\mathrm{Hom}_A(C_L,N)$ where N is the right (A,L)-module C_L, that is to say, $D_{\{\cdot,\cdot\}}$ satisfies Poincaré duality for just A, with its canonical left(A,L)-module structure, as asserted. This proves the claim. □

Example 7.17. Let $\mathfrak g$ be a real n-dimensional Lie algebra, consider its dual $\mathfrak g^*$, endowed with the ordinary Lie-Poisson structure, and write A for its Poisson algebra of smooth functions. The subalgebra of Casimir functions is the algebra $A^{\mathfrak g}$ of invariants, and we have the cup pairing

$$\mathrm{H}^k_{\mathrm{Poisson}}(A,A)\otimes_{A^{\mathfrak g}}\mathrm{H}^{n-k}_{\mathrm{Poisson}}(A,A)\longrightarrow \mathrm{H}^n_{\mathrm{Poisson}}(A,A). \tag{7.17.1}$$

Further, the Poisson cohomology $\mathrm{H}^*_{\mathrm{Poisson}}(A,A)$ is well known to be isomorphic to the Lie algebra cohomology $\mathrm{H}^*(\mathfrak g,A)$, the algebra A being endowed with the obvious $\mathfrak g$-module structure, cf. (3.18.4) in [15], and the pairing (7.17.1) boils down to the bilinear pairing

$$\mathrm{H}^k(\mathfrak g,A)\otimes_{A^{\mathfrak g}}\mathrm{H}^{n-k}(\mathfrak g,A)\longrightarrow \mathrm{H}^n(\mathfrak g,A). \tag{7.17.2}$$

In view of what has been said in (4.13), when $\mathfrak g$ is of compact type, the pairing (7.17.2) and hence (7.17.1) yields the nondegenerate bilinear pairing

$$\mathrm{H}^k_{\mathrm{Poisson}}(A,A)\otimes_{A^{\mathfrak g}}\mathrm{H}^{n-k}_{\mathrm{Poisson}}(A,A)\longrightarrow A^{\mathfrak g} \tag{7.17.3}$$

of finitely generated free $A^{\mathfrak g}$-modules.

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